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# Free objects and equational deduction for partial conditional specifications[☆]

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**Abstract**

Partial conditional specifications consist of conditional axioms, with equalities in the (possibly infinite set of) premises and in the consequence, which are interpreted in partial algebras.

Equalities may be existential ($=_e$) or strong ($=$); $t =_e t'$ holds in an algebra iff both $t$ and $t'$ are defined and equal, while $t = t'$ holds in an algebra iff either $t =_e t'$ holds or both $t$ and $t'$ are undefined. Contrary to the well explored case of positive conditional axioms (only existential equalities in the premises), general partial conditional specifications do not always admit free models and the related theory is much more subtle.

In this paper we fully investigate and solve the problem of existence of free and initial models, giving necessary and sufficient conditions, first from a model-theoretical and then from a logical deduction viewpoint.

In particular we present a deduction system which is complete w.r.t. strong equalities between open terms. Since positive conditional partial specifications and conditional total specifications are special cases of the paradigm investigated here, the presented theory generalizes the related results about free models and the Birkhoff-like deduction theory. The system we exhibit handles also the case of infinitary conjunctions as premises of the axioms; it reduces to a classical one for the positive conditional case by just dropping one rule, and finally it solves the empty-carrier problem without using explicit quantification.

The theory presented here also gives the basis for solving, via the usual first-order reduction, the problem of the existence of free and initial models for partial higher-order specifications of term-generated extensional models.

## 0. Introduction

Partial structures arise naturally in (mathematics and) computer science, either related to error/exception handling or to nontermination and this explains their

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intensive use and investigation (see e.g. [7–9, 18] for basic studies; [4, 5, 8] for applications). In particular, according to the initial algebra doctrine (see [15] for motivations and references), we are interested in algebraic specifications having (free and) initial models.

In computer science a special role is played by reachable (or term-generated) models, where every element can be denoted by a term; for example in [20] abstract data types are defined as isomorphism classes of reachable algebras, called computation structures there.

In the total algebra framework conditional specifications (with axioms of the form $\Delta \supset \varepsilon$, where $\varepsilon$ is an equation and $\Delta$ is a possibly infinite set of equalities) are the largest class always admitting term-generated initial models, which are initial both in the class of all models and of all term-generated models (see e.g. [19]). In the partial framework the existence of a (term-generated) initial model is always guaranteed only for positive conditional specifications, where in a conditional axiom $\Delta \supset \varepsilon$ every equality in $\Delta$ is existential (an existential equality of the form $t =_e t'$ holds in an algebra iff both $t$ and $t'$ are defined and equal). In the case of positive conditional specifications, the theory of the total case generalizes nicely (see [8]), both for the model-theoretical and for the logical aspects (equational deduction). Unfortunately the situation is much more subtle for general conditional equations, where the equational premises of an axiom may be strong (a strong equality of the form $t = t'$ holds in an algebra iff either $t =_e t'$ holds or both $t$ and $t'$ are undefined). Indeed in this case the model classes are not closed under products in general, the free model does not always exist, it is not straightforward to get a complete system of equational deduction and moreover the proof of completeness cannot be reduced to the existence of a free model.

Nonpositive conditional specifications arise naturally, at least implicitly, when dealing with specifications of partial functions, especially in a higher-order framework, which is now becoming a rather popular and useful tool in algebraic specifications (see [16, 17]). The key point is extensionality: assuming for simplicity that we consider just the homogeneous case (one sort) and unary functions $f, g : s \to s$, an algebraic higher-order specification has to satisfy implicitly the extensionality axiom

$$(\forall x : s : f(x) = g(x)) \;\; \supset \;\; f = g. \tag{$\star$}$$

This axiom, denoting as usual by $T_\Sigma$ the ground terms over a signature $\Sigma$, reduces for term-generated models to

$$\{ f(t) = g(t) \mid t \in T_\Sigma \} \;\; \supset \;\; f = g, \tag{$\star\star$}$$

which is an infinitary (countable) conditional axiom.

The axiom ($\star\star$), called "term-extensionality axiom", characterizes the class of term-extensional algebras, which is smaller than the class of extensional algebras but includes all term-generated extensional models. Now as usual we may reduce a higher-order specification to a first-order one (see [17] for a similar approach), with the help of auxiliary "apply" functions, so that (⋆⋆) becomes

$$\{\text{apply}(f,t) = \text{apply}(g,t) \mid t \in T_\Sigma\} \;\supset\; f = g.$$

Here the important point is that in the partial case the equality apply $(f,t) =$ apply$(g,t)$ (equivalently $f(t) = g(t)$) is a strong equality. Thus higher-order conditional specifications for term-extensional models, even if apparently positive conditional, reduce to first-order nonpositive conditional specifications.

Because of this we have been led to investigate the existence of free initial models for general partial conditional specifications, with an application to higher-order specifications.

After introducing notations and basic results about partial algebras (Section 1), we consider partial conditional specifications from a model theoretic viewpoint in Section 2. In that section, after introducing the basic definitions, we study the properties of models of partial conditional specifications and in particular the characterization and existence of free and initial models. We present a full picture of the situation that can be summarized as follows.

Like in the total and the partial positive conditional case, model classes are closed w.r.t. subalgebras and isomorphisms (Proposition 2.3); thus free and initial models are characterized, whenever exist, by the usual construction as a (partial) quotient of the term algebra w.r.t. the intersection of the kernels of the evaluation homomorphisms into the models of the specification (Theorem 2.4). Closure w.r.t. (even binary) products may fail and free and initial models do not always exist (counterexamples are provided). The existence of free models is undecidable even for finite specifications (Theorem 2.9); the proof uses a result (Theorem 2.7) showing that free models exist if a set of conditional formulas, called naughty formulas, is empty. The well-known existence of free models in the positive conditional case follows as an immediate corollary of this result. Finally it is shown that partial conditional specifications are more expressive than total and positive conditional specifications.

In Section 3 we discuss the relationships between the existence of the free models and logical deduction. The central result (Theorem 3.6) characterizes the existence of free models in terms of completeness w.r.t. existential equalities of generic conditional inference systems. However, we start the section by illustrating the delicate issue of soundness. Goguen and Meseguer (see [15] for references), following a remark by Huet, have shown that the usual one-sorted inference system trivially adapted to the many-sorted case produces unsound deductions whenever empty carriers are allowed, and proposed adding explicit quantification to the formulas to avoid the problem. Another way of handling the empty-carrier problems for total algebras is in [14]. In our formalism we can eliminate the unsound deductions without introducing explicit quantification. Our solution is illustrated by an important example of inference system, the $UL$ system, which is shown to be sound (Theorem 3.3). The $UL$ system is introduced both for the sake of concreteness and also as a preliminary to a complete system to be introduced later.

As an application of our results we get completeness results w.r.t. existential equalities (but not w.r.t. strong equalities) for systems handling positive conditional specifications.

In Section 4 we face the problem of giving systems (sound and) complete w.r.t. all equalities. This section is entirely devoted to a presentation and proof of completeness of systems w.r.t. strong equalities. These systems are obtained from the $UL$ system of the previous section just by adding one crucial (and subtle) elimination rule. In the first subsection two systems are presented, one for dealing with equalities with variables ($CL$) and the other for dealing with ground equalities. Their completeness, which is by far the most difficult result of the paper (Theorems 4.5 and 4.6) is then proved in the second subsection. Finally, a simplified system for the finitary case (i.e. only finite sets of premises in the axioms), where the elimination rule assumes a very simple and intuitive form, is given in the third subsection, where its completeness is nontrivially derived from the completeness of the system for the infinitary case (Theorem 4.19).

We stress the fact that the well-known results about free models and equational deduction for positive conditional and total specifications are all deducible as easy instantiations of the theory presented here, which thus can be seen as a comprehensive theory of partial specifications.

The results in this paper can of course be applied, via first-order reduction as we have indicated, to partial higher-order specifications and term-extensional (also term-generated extensional) models. In particular we can obtain necessary and sufficient conditions for the existence of the initial model in the class of term-extensional models for positive conditional higher-order specifications (see [1, 3]). Moreover we can get, via skolemization, complete systems of equational deduction for partial higher-order specifications and extensional models. We do not treat these applications here but refer to [1] and to [2] for some of them and to [3] for a full presentation of the higher-order case.

Some of the results of this paper in a less general form and without proofs already appeared in two conferences (see [1, 10]).

## 1. A category of partial algebras

We start with a short collection of basic notions and results, which are well known. However we need to report them here, in order to fix the notation and also because sometimes there are subtle differences; for example, the notion of congruence differs from the one in [9]. The notation here coincides more or less with that used by Meseguer and Goguen [15] and Broy and Wirsing in [8].

Proofs are omitted or simply outlined since they are straightforward adaptation of well known proofs for total algebras (see anyway [9, 15, 18]).

Due to the partiality of functions at the semantic level, metaexpressions can be "undefined", i.e. nondenoting any element; thus the meaning of equality between metaexpressions has to be explicitly stated, as different choices are possible. In the sequel the symbol $=$ will always denote strong equality, i.e. if $p$ and $q$ are expressions in the metalanguage, then $p = q$ holds iff either both $p$ and $q$ are undefined, or both are defined and equal.

**Definition 1.1.**

- A *signature* $(S, F)$ consists of a countable set $S$ of *sorts* and of a family $F = \{F_{w,s}\}_{w \in S^*, s \in S}$ of disjoint sets of *operation symbols*. We also write $op: s_1 \ldots s_n \to s$ for $op \in F_{s_1 \ldots s_n, s}$. A generic signature will be denoted by $\Sigma$.
- A *partial algebra* $A$ on a signature $\Sigma = (S, F)$ consists of a family $\{s^A\}_{s \in S}$ of sets, the *carriers*, and of a family $\{op^A\}_{op \in F_{w,s}, w \in S^*, s \in S}$ of partial functions, the *interpretations of operation symbols*, s.t.
  - if $w = \Lambda$ then either $op^A$ is undefined or $op^A \in s^A$;
  - if $w = s_1 \ldots s_n$, where $n \geqslant 1$, then $op^A: s_1^A \times \cdots \times s_n^A \to s^A$.

  For the case when $w = \Lambda$, we assume as usual that $\Lambda^A$ is the one point set and thus a constant $op \in F_{\Lambda, s}$, which is regarded as a "zeroary" operation, is interpreted in $A$ either as undefined or as a value in the codomain $s^A$.

  Often we denote the partial algebra $A$ by the pair $(\{s^A\}, \{op^A\})$, omitting the range of $s$ and $op$ which are given by the signature. A partial algebra over a signature $\Sigma$ is called a $\Sigma$-algebra. We denote by $PA(\Sigma)$ the class of all $\Sigma$-algebras.
- Let $A$ be a partial algebra on a signature $\Sigma = (S, F)$; then a $\Sigma$-algebra $B$ is a *subalgebra* (regular subobject) of $A$ iff $s^B \subseteq s^A$ for all $s \in S$ and for every $op \in F_{w,s}$
  - if $w = \Lambda$ then $op^B = op^A$;
  - if $w = s_1 \ldots s_n$, where $n \geqslant 1$, then $op^B(b_1, \ldots, b_n) = op^A(b_1, \ldots, b_n)$ for every $b_i \in s_i^B$ and $i = 1, \ldots, n$.

Note that if $B$ is a subalgebra of $A$ and $op^A(b_1, \ldots, b_n)$ is defined for some $b_i \in s_i^B$ and $i = 1, \ldots, n$, then $op^A(b_1, \ldots, b_n) \in s^B$, by definition of $op^B$. Thus subalgebras are operationally closed w.r.t. the interpretation of operation symbols in the algebra.

A particular example of algebra is the *term-algebra*, defined in the usual (total) way. In the sequel for every signature $\Sigma = (S, F)$ we assume fixed a *universe* $Var = \{Var_s\}_{s \in S}$ of variables, i.e. an $S$-sorted family of disjoint denumberable sets, so that every "family $X$ of variables" stands for a "subfamily $X$ of $Var$".

**Definition 1.2.** Let $\Sigma = (S, F)$ be a signature and $X = \{X_s\}_{s \in S}$ be a family of variables.

- The family $\{T_\Sigma(X)_{|s}\}_{s \in S}$ of the *sets of terms* is inductively defined by:
  - for all $s \in S$ we have $X_s \cup F_{\Lambda, s} \subseteq T_\Sigma(X)_{|s}$;
  - for all $op \in F_{s_1 \ldots s_n, s}$ and all $t_i \in T_\Sigma(X)_{|s_i}$ for $i = 1, \ldots, n$ we have $op(t_1, \ldots, t_n) \in T_\Sigma(X)_{|s}$.
- For all $op \in F_{s_1 \ldots s_n, s}$ let $op^T: T_\Sigma(X)_{|s_1} \times \cdots \times T_\Sigma(X)_{|s_n} \to T_\Sigma(X)_{|s}$ be defined by $op^T(t_1, \ldots, t_n) = op(t_1, \ldots, t_n)$ for all $t_i \in T_\Sigma(X)_{|s_i}$ for $i = 1, \ldots, n$.
- The *term algebra* over $\Sigma$ and $X$, denoted by $T_\Sigma(X)$, or shortly $T_\Sigma$ if $X$ is the (family of) empty set, is the pair $(\{T_\Sigma(X)_{|s}\}, \{op^T\})$.

The usual homomorphism condition imposed on homomorphisms $p: A \to B$

$$p_s(op^A(a_1, \ldots, a_n)) = op^B(p_{s_1}(a_1), \ldots, p_{s_n}(a_n)) \qquad (\star)$$

can be relaxed in the partial paradigm from two points of view: Eq. $(\star)$ can be required to be satisfied only on those values $a_i$ s.t. $op^A(a_1, \ldots, a_n)$ (respectively $op^B(p_{s_1}(a_1), \ldots, p_{s_n}(a_n))$) is defined and moreover $p$ can be a partial function. This gives rise to 6 different notions of homomorphism (all composable and with identity) that are used in the literature for different purposes, see e.g. [8, 9, 18].

Since in the sequel we will be primarily interested in initial models, we will define homomorphisms (and the corresponding category) in a way that the initial model (if any) satisfies the "no-junk" and "no-confusion" properties, see [15]. These are guaranteed if we require, respectively, that $p$ is total and that $(\star)$ holds if $op^A(a_1, \ldots, a_n)$ is defined.

**Definition 1.3.** Let $A$ and $B$ be two $\Sigma$-algebras and $p$ be a family of total functions $p = \{p_s : s^A \to s^B\}_{s \in S}$. Then $p$ is a *homomorphism* from $A$ into $B$ iff for any $op \in F_{s_1 \ldots s_n, s}$, with $n \geq 0$, and any, $a_i \in s_i^A$ with $i = 1, \ldots, n$,

$$op^A(a_1, \ldots, a_n) \in s^A \text{ implies } p_s(op^A(a_1, \ldots, a_n)) = op^B(p_{s_1}(a_1), \ldots, p_{s_n}(a_n)).$$

In the sequel if $p$ is a homomorphism from $A$ into $B$, then we write $p: A \to B$.

Note that the homomorphisms are composable (as maps) and that the identity map is always a homomorphism; thus we can define a category $\mathbf{PAlg}(\Sigma)$ having $\Sigma$-algebras as objects and homomorphisms as morphisms. In the following definition we use the notation of [13].

**Definition 1.4.** For each signature $\Sigma = (S, F)$ the category $\mathbf{PAlg}(\Sigma)$ of partial $\Sigma$-algebras consists of:

- the set of objects of $\mathbf{PAlg}(\Sigma)$ is $PA(\Sigma)$;
- $\mathbf{PAlg}(\Sigma)(A, B) = \{p: A \to B \mid p \text{ homomorphism}\}$ for all $A, B \in PA(\Sigma)$;
- the *identity* morphism $Id_A$ is $Id_A = \{Id_{s^A}\}_{s \in S}$, where $Id_{s^A}$ is defined by $Id_{s^A}(a) = a$ for all $a \in s^A$ and all $\Sigma$-algebras $A$;
- let $A$, $B$ and $C$ be $\Sigma$-algebras, $p: A \to B$ and $q: B \to C$ be homomorphisms; then $q \cdot p$ is the family $\{q_s \cdot p_s\}_{s \in S}$.

**Definition 1.5.** Let $A$ be a $\Sigma$-algebra, $X = \{X_s\}_{s \in S}$ be an $S$-sorted family of variables and $V = \{V_s : X_s \to s^A\}_{s \in S}$ be a family of total functions, called a *valuation* for $X$ in $A$.

- Then the *natural interpretation* of terms w.r.t. $A$ and $V$, denoted by $eval^{A,V}$, is the partial function inductively defined by the following clauses, where we write $t^{A,V}$ for $eval^{A,V}(t)$:
  - $x^{A,V} = V_s(x)$, for all $x \in X_s$ and $op^{A,V} = op^A$, for all $op \in F_{\Lambda,s}$;
  - $(op(t_1, \ldots, t_n))^{A,V} = op^A(t_1^{A,V}, \ldots, t_n^{A,V})$ for all $op \in F_{s_1 \ldots s_n, s}$, with $n \geq 1$, and all $t_i \in T_\Sigma(X)_{|s_i}$.

When restricted to $T_\Sigma$, $eval^{A,V}$ is denoted by $eval^A$ and, correspondingly, $t^{A,V}$ becomes $t^A$.

- The *term-image* algebra $V(A)$ is the subalgebra of $A$ defined by:

$$s^{V(A)} = \{a \mid \exists t \in T_\Sigma(X)_{|s} \text{ s.t. } a = t^{A,V}\} \text{ for all } s \in S.$$

- The *kernel* of the natural interpretation of terms w.r.t. $A$ and $V$, denoted by $K^{A,V}(X)$ or simply by $K^A$ if $X$ is the empty set, is the family $\{K^{A,V}(X)_s\}_{s\in S}$ where

$$K^{A,V}(X)_s = \{(t,t') \mid t,t' \in T_\Sigma(X)_{|s},\ t^{A,V},\ t'^{A,V} \in s^A \text{ and } t^{A,V} = t'^{A,V}\}.$$

- If $eval^A$ is surjective, then $A$ is called *term-generated.*

**Proposition 1.6.** *Let $A$ and $B$ be two $\Sigma$-algebras, $X$ be a family of variables, $V_A$ and $V_B$ be two valuations for $X$ in $A$ and $B$ respectively.*

1. *If $p: A \to B$ is a homomorphism s.t. $p(V_A(x)) = V_B(x)$ for all $x \in X$, then $p(t^{A,V_A}) = t^{B,V_B}$ for all $t \in T_\Sigma(X)_{|s}$ s.t. $t^{A,V_A} \in s^A$.*
2. *If $eval^{A,V_A}$ is surjective, then there exists at the most one homomorphism $q: A \to B$ s.t. $q(V_A(x)) = V_B(x)$ for all $x \in X$.*

**Proof.** By induction over $T_\Sigma(X)$. □

**Definition 1.7.**

- Given a signature $\Sigma = (F,S)$ and a $\Sigma$-algebra $A$, a *congruence* $\equiv$ over $A$ is a family of binary relations $\{\equiv_s\}_{s\in S}$ satisfying the following conditions (where we omit the obvious quantifications over sorts):
  - $\equiv_s \subseteq s^A \times s^A$, and $\equiv_s$ is symmetric and transitive; in the sequel we denote by $Dom(\equiv_s)$ the set $\{a \mid (a,a) \in \equiv_s\}$ and we define $a \equiv_s^D a'$ iff either $a \equiv_s a'$ or $a, a' \notin Dom(\equiv_s)$;
  - for all $op \in F_{s_1\ldots s_n,s}$ and all $a_i, a_i' \in s_i^A$, with $i = 1,\ldots,n$, if $a_i \equiv_{s_i} a_i'$ for $i = 1,\ldots,n$, then $op^A(a_1,\ldots,a_n) \equiv_s^D op^A(a_1',\ldots,a_n')$.
  - for all $op \in F_{s_1\ldots s_n,s}$ and all $a_i \in s_i^A$, with $i = 1,\ldots,n$, if $op^A(a_1,\ldots,a_n) \in Dom(\equiv_s)$, then $a_i \in Dom(\equiv_{s_i})$, for $i = 1,\ldots,n$.
- Let $\equiv$ be a congruence over a $\Sigma$-algebra $A$; let $[a]$ denote the equivalence class of $a$ in $\equiv_s$ for all $s \in S$ and all $a \in s^A$. The *quotient algebra* of $A$ w.r.t. $\equiv$, denoted by $A/\equiv$, is defined by:
  - $s^{A/\equiv} = \{[a] \mid a \in Dom(\equiv_s)\}$, for all $s \in S$;
  - $op^{A/\equiv}([a_1],\ldots,[a_n]) = [op^A(a_1,\ldots,a_n)]$ if $op^A(a_1,\ldots,a_n) \in Dom(\equiv_s)$, otherwise $op^{A/\equiv}([a_1],\ldots,[a_n])$ is undefined, for all $op \in F_{s_1\ldots s_n,s}$, $a_i \in Dom(\equiv_{s_i})$ with $i = 1,\ldots,n$.

The definition of congruence guarantees that structured denotations of elements are preserved by the quotient operation.

**Proposition 1.8.** *Let $A$ be a $\Sigma$-algebra, $\equiv$ be a congruence, $X$ be a family of variables and $V$, $V'$ be valuations respectively for $X$ in $A$ and for $X$ in $A/\equiv$ s.t. $V'(x) = [V(x)]$. For every term $t \in T_\Sigma(X)_{|s}$ we have that $[t^{A,V}] = t^{A/\equiv,V'}$.*

**Proof.** By structural induction over $T_\Sigma(X)$. $\square$

The Proposition 1.8 implies, in particular, that if $eval^{A,V}$ is surjective, then $eval^{A/\equiv,V'}$ is surjective, too.

**Proposition 1.9.** *Let $A$ be a $\Sigma$-algebra and $V$ be a valuation for a family $X$ of variables in $A$.*
1. *The kernel $K^{A,V}(X)$ is a congruence over $T_\Sigma(X)$.*
2. *The algebras $V(A)$ and $T_\Sigma(X)/K^{A,V}(X)$ are isomorphic.*

**Proof.**
1. The proof easily follows from the definition of $K^{A,V}(X)$.
2. It is easy to check that $p: T_\Sigma(X)/K^{A,V}(X) \to V(A)$, defined by $p_s([t]) = t^{A,V}$, and $q: V(A) \to T_\Sigma(X)/K^{A,V}(X)$, defined by $q_s(a) = [t]$, where $t \in T_\Sigma(X)_{|s}$ and $t^{A,V} = a$, are homomorphisms and that both $p \cdot q = Id_{V(A)}$ and $q \cdot p = Id_{T_\Sigma(X)/K^{A,V}(X)}$. $\square$

**Proposition 1.10.** *Let $A$ be a $\Sigma$-algebra and $\equiv_1$, $\equiv_2$ be congruences over $A$. Then the relation $M \subseteq A/\equiv_1 \times A/\equiv_2$, defined by $M = \{([a]_1, [a]_2) \mid a \in Dom(\equiv_1) \cup Dom(\equiv_1)\}$, is a homomorphism iff $\equiv_1 \subseteq \equiv_2$.*

**Proof.** The proof easily follows from the definitions of congruence and homomorphism. $\square$

It is easy to check that the intersection of congruences is a congruence, too; thus we can give the following definition.

**Definition 1.11.** Let $C$ be a class of $\Sigma$-algebra and $X$ be a family of variables s.t. there exist an $A \in C$ and a valuation for $X$ in $A$.
- For every family $\equiv\, = \{\equiv^i\}_{i \in I}$ of congruences over a $\Sigma$-algebra $A$ the *intersection of* $\equiv$, denoted by $\cap(\equiv)$, is the congruence $\{\bigcap_{i \in I} \equiv^i_s\}_{s \in S}$.
- $K^C(X)$ is the intersection of the family $\{K^{A,V}(X) \mid A \in C,\ V: X \to A\}$. If $X$ is the empty set, then we simply denote $K^C(X)$ by $K^C$. Moreover we denote $T_\Sigma(X)/K^C(X)$ by $Fr^C(X)$ and by $m^C$ the valuation $m^C: X \to Fr^C(X)$ defined by $m^C(x) = [x]_{K^C(X)}$.
- $Gen(C, X)$ is the subclass of $C$ defined by:

  $\{A \mid A \in C$ and there exists $V: X \to A$ s.t. $V(A) = A\}$.

  If $X = \emptyset$, then $Gen(C, \emptyset) = \{A \mid A \in C$ s.t. $eval^A(T_\Sigma) = A\}$ is denoted by $Gen(C)$.
- A pair $(Fr, m)$, where $Fr$ is a $\Sigma$-algebra and $m$ is a valuation for $X$ in $Fr$, is *free* over $X$ in $C$ iff
  - $Fr \in C$;

- for all $A \in C$ and all valuations $V$ for $X$ in $A$, there exists a unique homomorphism $p_V$ from $Fr$ into $A$ s.t. $p_V(m(x)) = V(x)$ for all $x \in X$.

- An algebra $I$ is initial in $C$ iff it is free over the empty set in $C$, i.e. iff $I \in C$ and for all $A \in C$ there exists a unique homomorphism from $I$ into $A$.

Note that the definition of free object coincides with the usual definition in category theory of a free object generated by $X$ w.r.t. the forgetful functor from $\mathbf{PAlg}(\Sigma)$ into $\mathbf{Set}_S$ (the category of $S$-sorted sets), see e.g. [13].

Let us state some results on the existence and the characterization of the free model for a class $C$ of algebras.

**Proposition 1.12.** *Let $X$ be a family of variables and $C$ be a class of $\Sigma$-algebras closed w.r.t. subalgebras and isomorphisms. The following conditions are equivalent:*
1. *there exists a free object for $X$ in $C$;*
2. *$Fr^C(X)$ belongs to $C$;*
3. *$(Fr^C(X), m^C)$ is free for $X$ in $C$;*
4. *there exists a free object for $X$ in $Gen(C, X)$.*

**Proof.** $(1 \rightarrow 2)$ Let $(Fr, m)$ be the free object for $X$ in $C$. Since $C$ is closed w.r.t. subalgebras, $m(Fr) \in C$ and hence $T_\Sigma(X)/K^{Fr,m}(X) \in C$, too, because of Proposition 1.9, as $C$ is closed w.r.t. isomorphisms. Since $(Fr, m)$ is free for $X$ in $C$, for every $A \in C$ and every $V: X \rightarrow A$ a homomorphism $q_V: Fr \rightarrow A$ exists s.t. $q_V \cdot m = V$, so that, by Proposition 1.6, $q_V(t^{Fr,m}) = t^{A,V}$ for all $t \in T_\Sigma(X)$ s.t. $t^{Fr,m}$ is defined.

Therefore $p_V: T_\Sigma(X)/K^{Fr,m}(X) \rightarrow T_\Sigma(X)/K^{A,V}(X)$, defined by $p_V([t]_{K^{Fr,m}(X)}) = [t]_{K^{A,V}(X)}$ is a homomorphism. Thus $K^{Fr,m}(X) \subseteq K^{A,V}(X)$ because of Proposition 1.10, and hence $K^{Fr,m}(X) \subseteq K^C(X)$ so that $K^{Fr,m}(X) = K^C(X)$. Therefore $Fr^C(X) = T_\Sigma(X)/K^C(X) = T_\Sigma(X)/K^{Fr,m}(X) \in C$.

$(2 \rightarrow 3)$ We have to show that for all $A \in C$ and all $V: X \rightarrow A$ there exists a unique homomorphism $p_V: Fr^C(X) \rightarrow A$ s.t. $p_V([x]) = V(x)$. Let $p_V: Fr^C(X) \rightarrow A$ defined by $p_V([t]) = t^{A,V}$; $p_V$ is a function because $K^C(X) \subseteq K^{A,V}(X)$ and obviously is a homomorphism. Finally $p_V$ is unique because of Proposition 1.6.

$(3 \rightarrow 4)$ Since $Fr^C(X) \in Gen(C, X)$ by definition and $(Fr^C(X), m^C)$ is free for $X$ in $C$, then $(Fr^C(X), m^C)$ is also free for $X$ in $Gen(C, X)$.

$(4 \rightarrow 1)$ Let $(Fr, m)$ be the free object for $X$ in $Gen(C, X)$. Consider $A \in C$ and $V: X \rightarrow A$. Then $V(A) \in Gen(C, X)$, hence there is $p: Fr \rightarrow V(A)$ such that $p(m(x)) = V(x)$ for all $x \in X$. Moreover the composition of such $p$ with the embedding of $V(A)$ into $A$ is the unique morphism from $Fr$ to $A$ satisfying the above condition, by Proposition 1.6. □

**Corollary 1.13.** *Let $C$ be a class of $\Sigma$-algebras closed w.r.t. subalgebras and isomorphisms. The following conditions are equivalent:*
1. *there exists an initial object in $C$;*
2. *$T_\Sigma/K^C$ belongs to $C$;*

3. *$T_\Sigma/K^C$ is initial in $C$;*
4. *there exists a free object for $X$ in $Gen(C)$.*

**Proof.** From Proposition 1.12, for $X = \emptyset$. □

## 2. Partial conditional specifications

In this section, after introducing some basic definitions, we study the properties of models of partial conditional specifications and in particular the characterization and existence of free and initial models.

We present a full picture of the situation that can be summarized as follows.

Like in the total and the partial positive conditional case, model classes are closed w.r.t. subalgebras and isomorphisms (Proposition 2.3); thus free and initial models are characterized, whenever exist, by the usual construction as a (partial) quotient of the term algebra w.r.t. the intersection of the kernels of the evaluation homomorphisms into the models of the specification (Theorem 2.4).

Closure w.r.t. (even binary) products may fail and free and initial models do not always exist (counterexamples are provided).

The existence of free models is undecidable even for finite specifications (Theorem 2.9); the proof of this uses a result (Theorem 2.7) showing that a necessary and sufficient condition for the existence of free models is the emptiness of a set of conditional formulas, called naughty formulas. This also shows the well-known existence of free models in the positive conditional case.

Finally it is shown that partial conditional specifications are more expressive than total and positive conditional specifications.

**Definition 2.1.** Let $\Sigma = (S, F)$ be a signature and $X$ be a family of $S$-sorted variables.

- An *elementary formula* over $\Sigma$ and $X$ has the form either $D(t)$ or $t = t'$ for $t, t' \in T_\Sigma(X)_{|s}$, where $D$ denotes the definedness predicate (one for each sort; but sorts are omitted). The set of all elementary formulas over $\Sigma$ and $X$ will be denoted by EForm $(\Sigma, X)$.
- A *conditional formula* over $\Sigma$ and $X$ has the form $\Delta \supset \varepsilon$, where $\Delta$ and $\varepsilon$ are respectively a countable set of elementary formulas and an elementary formula over $\Sigma$ and $X$.

  If $\Delta$ is the empty set, then $\Delta \supset \varepsilon$ is an equivalent notation for the elementary formula $\varepsilon$.
- A *positive conditional formula* over $\Sigma$ and $X$ is a conditional formula $\Delta \supset \varepsilon$ over $\Sigma$ and $X$ s.t. $D(t)$ or $D(t')$ belongs to $\Delta$ for every $t = t'$ belonging to $\Delta$.
- For every formula $\phi$ let $Var(\phi)$ denote the set of all variables which appear in $\phi$. A formula $\phi$ is called *ground* iff $Var(\phi)$ is empty.

- If $A$ is a partial algebra, $\phi$ is a formula and $V$ is a valuation for $Var(\phi)$ in $A$, then we say that $\phi$ *holds* for $V$ in $A$ (equivalently: *is satisfied for* $V$ *by* $A$) and write $A \models_V \phi$ accordingly to the following clauses:
  - $A \models_V D(t)$ iff $t^{A,V}$ is defined;
  - $A \models_V t = t'$ iff $t^{A,V}$, $t'^{A,V}$ are either both defined and equal or both undefined;
  - $A \models_V \Delta \supset \varepsilon$ iff $A \models_V \varepsilon$, or $A \not\models_V \delta$ for some $\delta \in \Delta$;

  We write $A \models \phi$ for a formula $\phi$ and say that $\phi$ *holds in* (equivalently: *is satisfied by, is valid in*) $A$ iff $A \models_V \phi$ for all valuations $V$ for $Var(\phi)$ in $A$.

**Remark**

- From the definition of validity, for all variables $x$ we have that $A \models D(x)$ follows, because valuations are total functions and thus $V(x) = x^{A,V}$ is defined for every valuation $V$.
- Note that $D(t)$ can be equivalently expressed by $t =_e t$, where $=_e$ denotes existential equality: $=_e$ holds iff both sides are defined and equal; hence elementary formulas are just equalities either strong or existential. Analogously, since $t =_e t'$ is logically equivalent to $D(t) \wedge t = t'$, positive conditional formulas are (first-order equivalent to) conditional formulas whose premises are just existential equalities.
- The above notion of validity is the usual one in the many-sorted case; however some comments can be helpful. If $Var(\phi)_s \neq \emptyset$ and $s^A = \emptyset$, then $A \models \phi$ holds; hence for any class $C$ of algebras, $C \models \phi$ iff $A \models \phi$ for all $A \in C$ s.t. $Var(\phi)_s \neq \emptyset$ implies $s^A \neq \emptyset$. Thus if $C$ contains an algebra with all carriers nonempty (as it will always happen in the sequel), then the notion of validity for the class coincides with the classical one; for example we could not have both $C \models \phi$ and $C \models \neg\phi$ (but note that here we do not have negation). Finally it is also useful to emphasize that here we can stay within a two-valued logic, since any conditional formula is always either true or false for a (total) valuation of its variables.

In the sequel a generic elementary formula will be denoted by $\varepsilon$, $\eta$, $\gamma$ or $\delta$, while a generic conditional formula will be denoted by $\phi$, $\theta$ or $\psi$; moreover for all conditional formulas $\phi = (\Delta \supset \varepsilon)$ we denote $\Delta$ by prem($\phi$) and $\varepsilon$ by cons($\phi$); finally $\varepsilon_1 \wedge \cdots \wedge \varepsilon_n \supset \varepsilon$ is the same as $\{\varepsilon_1, \ldots, \varepsilon_n\} \supset \varepsilon$ for elementary formulas $\varepsilon_1, \ldots, \varepsilon_n, \varepsilon$.

**Definition 2.2**

- A *conditional specification* consists of a signature $\Sigma$ and of a set $Ax$ of conditional formulas over $\Sigma$. A generic conditional specification will be denoted by $Sp$; the formulas belonging to $Ax$ are called the *axioms* of $Sp$ and usually denoted by $\alpha$.
- A *positive conditional specification* is a conditional specification s.t. all its axioms are positive conditional formulas; a generic positive conditional specification will be denoted by $PSp$.

- For any conditional specification $Sp = (\Sigma, Ax)$,

  $$PMod(Sp) = \{A \mid A \in PA(\Sigma) \text{ and } A \models \alpha \text{ for all } \alpha \in Ax\};$$

  an algebra $A \in PMod(Sp)$ is called a *model* of $Sp$.
- For every conditional specification $Sp = (\Sigma, Ax)$ and every family $X$ of variables, $K(Sp, X)$ denotes the congruence $K^{PMod(Sp)}(X)$, $Fr(Sp, X)$ denotes $Fr^{PMod(Sp)}(X)$ and $m(Sp, X)$ denotes the valuation $m^{PMod(Sp)}$ (see Definition 1.11).
- For every conditional specification $Sp = (\Sigma, Ax)$, $PGen(Sp, X)$ denotes the class $Gen(PMod(Sp), X)$; moreover, if $X$ is empty, then $PGen(Sp, X)$ is simply denoted by $PGen(Sp)$.

Note that $PMod(Sp)$ is not empty for any conditional specification $Sp$, since the trivial (total) algebra $Z$, with singleton sets as carriers and the obvious (total) interpretations of function symbols, is always a model. Moreover the trivial algebra $Z$ has all carriers nonempty, so that there exists a valuation for all families $X$ in $Z$ and hence $K(Sp, X)$ is always well defined.

**Proposition 2.3.** *For all conditional specifications $Sp$ the class $PMod(Sp)$ is closed w.r.t. subalgebras and isomorphisms.*

**Proof.** The closure under isomorphisms easily follows from the definition of validity; thus we just consider the closure under subobjects.

Let $A$ belong to $PMod(Sp)$, $B$ be a subalgebra of $A$, $\alpha$ be an axiom and $V$ be a valuation for $Var(\alpha)$ into $B$. Then $V$ is also a valuation for $Var(\alpha)$ into $A$ and it is easy to check that $t^{A,V} = t^{B,V}$ for all $t \in T_\Sigma(Var(\alpha))$ and hence that $A \models_V \varepsilon$ iff $B \models_V \varepsilon$ for all $\varepsilon \in \mathrm{EForm}(\Sigma, Var(\alpha))$. Therefore $B \models_V \alpha$ follows, since $A \models_V \alpha$, because $A$ is a model of $Sp$. □

Thus we can instantiate Proposition 1.12 on $PMod(Sp)$.

**Theorem 2.4.** *Let $X$ be a family of variables and $Sp = (\Sigma, Ax)$ be a conditional specification. The following conditions are equivalent:*
1. *there exists a free object for $X$ in $PMod(Sp)$;*
2. *$Fr(Sp, X) \in PMod(Sp)$;*
3. *$(Fr(Sp, X), m(Sp, X))$ is the free object for $X$ in $PMod(Sp)$;*
4. *there exists a free object for $X$ in $PGen(Sp, X)$.*

**Proof.** From Proposition 1.12, which applies here because $PMod(Sp)$ is closed w.r.t. subalgebras and isomorphisms by Proposition 2.3. □

Contrary to the case of positive conditional specifications, in general the class of models of a conditional specification need not be closed under binary products, as the following example shows.

**spec** $Sp_1$ =
  **sorts** $s$
  **opns**
    $a, b : \rightarrow s$
  **axioms**
    $a = b \supset D(a)$

Let $A$ and $B$ be the models of $Sp_1$ defined by:

**Algebra** $A$ =
  $s^A = \{1\}$
  $a^A$ is undefined
  $b^A = 1$

**Algebra** $B$ =
  $s^B = s^A$
  $a^B = 1$
  $b^B$ is undefined

Then the algebra $A \times B$ consists of:

**Algebra** $A \times B$ =
  $s^{A \times B} = \{(1, 1)\}$
  $a^{A \times B} = (a^A, a^B)$ is undefined, because $a^A$ is undefined
  $b^{A \times B} = (b^A, b^B)$ is undefined, because $b^B$ is undefined

Therefore $A \times B$ is not a model of $Sp_1$, because $A \times B \models a = b$, as both $a$ and $b$ are undefined, but $A \times B \not\models D(a)$. □

While in the case of positive conditional specifications the closure under isomorphisms, subalgebras and products is sufficient to guarantee the existence of (free and) initial objects, the model class of a conditional specification does not need to have initial objects. Indeed, let us consider again the above specification $Sp_1$; since an initial model, if any, is minimally defined, if an algebra $I$ is initial in $PMod(Sp_1)$, then both $a$ and $b$ are undefined, because they are undefined respectively in $A$ and $B$, and hence $I$ is not a model of $Sp_1$. It is easy to see that more sophisticated specifications exist that admit initial models, but do not admit free models for nonempty $X$; consider indeed the following specification $Sp_2$.

**spec** $Sp_2$ =
  **sorts** $s$
  **opns**
    $zero : \rightarrow s$
    $f, Succ : s \rightarrow s$

**axioms**

$\alpha_1$ $Succ(x) = f(x) \supset D(Succ(x))$

$\alpha_2$ $D(Succ(zero))$

$\alpha_3$ $D(Succ(x)) \supset D(Succ(Succ(x)))$

The following is an initial model of $Sp_2$:

**Algebra** $I =$

$s^I = \mathbb{N}$

$zero^I = 0$

$Succ^I(a) = a + 1$ $f^I$ is the totally undefined function

Let $A$ and $B$ be the models of $Sp_2$ and $V_A$, $V_B$ the valuations for a nonempty $X$ in $A$, $B$ respectively defined by:

**Algebra** $A =$

$s^A = \mathbb{N} \cup \{\infty\}$

$zero^A = 0$

$Succ^A(a) = a + 1$ if $a \in \mathbb{N}$

$Succ^A\{\infty\}$ is undefined

$f^A(a) = \infty$

$V_A(x) = \infty$ for all $x \in X$

**Algebra** $B =$

$s^B = s^A$

$zero^B = 0$

$Succ^B(b) = b + 1$ if $b \in \mathbb{N}$

$Succ^B\{\infty\} = \infty$

$f^B$ is the totally undefined function

$V_B = V_A$

Because of Theorem 2.4, in order to show that $Sp_2$ has no free model for $X$ it is sufficient to show that $Fr(Sp_2, X) \notin PMod(Sp_2)$. Since both $Succ(x)^{A,V_A}$ and $f(x)^{B,V_B}$ are undefined for all $x \in X$, they do not belong to $Dom(K(Sp_2, X))$. Therefore $Fr(Sp_2, X) \models_{m(Sp_2,X)} Succ(x) = f(x)$ and $Fr(Sp_2, X) \not\models_{m(Sp_2,X)} D(Succ(x))$ so that $Fr(Sp_2, X)$ does not satisfy $\alpha_1$ and hence is not a model of $Sp_2$. □

Since, by Theorem 2.4, the existence of a free model of a specification $Sp$ for a family $X$ of variables is equivalent to $Fr(Sp, X) \in PMod(Sp)$, we are interested in conditions guaranteeing that $Fr(Sp, X)$ satisfies the axioms of $Sp$. Since $Fr(Sp, X)$ is a quotient of a term algebra, such conditions can be stated in a syntactic form. Indeed, a quotient $T_\Sigma(X)/\equiv$ satisfies a formula $\phi$ for a valuation $V$ iff it satisfies an instantiation $\rho(\phi)$ for the valuation $m$ defined by $m(x) = [x]_\equiv$, where $\rho$ substitutes for each variable $x \in Var(\phi)$ a representative of the congruence class $V(x)$.

**Definition 2.5.** Let $Sp = (\Sigma, Ax)$ be a conditional specification, $X$ be a family of variables, $\equiv$ be a congruence on $T_\Sigma(X)$ and $m$ be the valuation defined by $m(x) = [x]_\equiv$. The set $\mathrm{NF}(Sp, \equiv)$, where NF stands for Naughty Formulas, consists of all conditional formulas $\Delta \supset \varepsilon$ over $\Sigma$ and $X$ that are instantiations of the axioms and do not hold in $T_\Sigma(X)/\equiv$, i.e. more formally s.t.

$nf_1$ $\Delta \supset \varepsilon$ is $\alpha[t_y/y \mid y \in Var(\alpha)]$ for some $\alpha \in Ax$ and some $t_y \in T_\Sigma(X)$ s.t. $t_y \in Dom(\equiv)$ for all $y \in Var(\alpha)$;

$nf_2$ $T_\Sigma(X)/\equiv \models_m \delta$ for all $\delta \in \Delta$;

$nf_3$ $T_\Sigma(X)/\equiv \not\models_m \varepsilon$.

In the special case when $\equiv = K(Sp, X)$, the set $\mathrm{NF}(Sp, K(Sp, X))$ will be denoted by $\mathrm{SNF}(Sp, X)$, where SNF stands for Semantic Naughty Formulas.

**Notation.** Let $X$ and $Y$ be two families of variables, $K$ be a congruence over $T_\Sigma(Y)$ and $V$ be a valuation for $X$ in $T_\Sigma(Y)/K$. For every $x \in X$ we denote by $t_{V,x}$ a term $t_{V,x} \in dom(K)$ s.t. $V(x) = [t_{V,x}]$, by $V(t)$ the term $t[t_{V,x}/x \mid x \in X]$ for every term $t$, by $V(\theta)$ the formula $\theta[t_{V,x}/x \mid x \in X]$ for every formula $\theta$ and by $V(\Gamma)$ the set $\{V(\gamma) \mid \gamma \in \Gamma\}$ for every set $\Gamma$ of formulas.

Note that if $A = T_\Sigma(Y)/K$, then $t^{A,V}$ is the equivalence class of $V(t)$ in $K$, by Proposition 1.8, and that $A \models_V \gamma$ iff $A \models_m V(\gamma)$, where $m(x) = [x]_K$.

**Lemma 2.6.** *Let $Sp = (\Sigma, Ax)$ be a conditional specification, $X$ be a family of variables and $\equiv$ be a congruence on $T_\Sigma(X)$; then $T_\Sigma(X)/\equiv$ is a model of $Sp$ iff* $\mathrm{NF}(Sp, \equiv) = \emptyset$.

**Proof.** Let $A$ denote $T_\Sigma(X)/\equiv$ and $m$ be the valuation defined by $m(x) = [x]_\equiv$ for all $x \in X$.

($\Rightarrow$) We assume that $\phi$ satisfies $nf_1$ and $nf_2$ and show that $\phi$ does not satisfy $nf_3$. Because of $nf_1$, $\phi$ is $\alpha[t_y/y \mid y \in Var(\alpha)]$ for some $\alpha \in Ax$ and $t_y \in T_\Sigma(X)$ s.t. $A \models_m D(t_y)$. Let us define the valuation $V$ for $Var(\alpha)$ in $A$ by $V(y) = [t_y]$; note that $V$ is well defined, because $A \models_m D(t_y)$ by $nf_1$ and hence $[t_y] \in A$. Since $A$ is a model of $Sp$, we have that $A \models_V \alpha$, i.e. $A \models_m V(\alpha)$, hence $A \models_m \phi$. Therefore, since $A \models_m \delta$ for all $\delta \in \mathrm{prem}(\phi)$ because of $nf_2$, $A \models_m \mathrm{cons}(\phi)$, i.e. $\phi$ does not satisfy $nf_3$.

($\Leftarrow$) Let $\alpha$ be an axiom of $Sp$ and $V$ be a valuation for $Var(\alpha)$ in $A$. Then for all $y \in Var(\alpha)$ we have that $t_{V,y} \in Dom(\equiv)$, i.e. $A \models_m D(t_{V,y})$ and hence $V(\alpha)$ satisfies $nf_1$. Thus, since $\mathrm{NF}(Sp, \equiv)$ is empty, $V(\alpha)$ does not satisfy $nf_2$ or $nf_3$, i.e. $A \not\models_m \delta$ for some $\delta \in \mathrm{prem}(V(\alpha))$ or $A \models_m \mathrm{cons}(V(\alpha))$. Thus $A \models_m V(\alpha)$ and hence $A \models_V \alpha$. $\square$

Adding the results of Theorem 2.4 to an application of Lemma 2.6 we get the following summarizing theorem, which will be used extensively in the sequel.

**Theorem 2.7.** *For every specification Sp and every family X of variables, the following conditions are equivalent*:

1. *there exists a free object for* $X$ *in* $PMod(Sp)$;
2. $Fr(Sp, X) \in PMod(Sp)$;
3. $(Fr(Sp, X), m(Sp, X))$ *is free for* $X$ *in* $PMod(Sp)$;
4. *there exists a free object for* $X$ *in* $PGen(Sp, X)$;
5. $\mathrm{SNF}(Sp, X) = \emptyset$.

**Proof.** $1 \Leftrightarrow 2 \Leftrightarrow 3 \Leftrightarrow 4$ follow from Proposition 2.4 and $2 \Leftrightarrow 5$ from Lemma 2.6. □

Using condition 5 of Theorem 2.7, we can obtain as a corollary the well-known result of the existence of a free model for positive conditional specifications.

**Corollary 2.8.** *If PSp is a positive conditional specification, then for all families X of variables* $(Fr(PSp, X), m(PSp, X))$ *is free for* $X$ *in* $PMod(PSp)$.

**Proof.** Let $Fr$ denote $Fr(PSp, X)$ and $m$ denote $m(PSp, X)$. Because of Theorem 2.7, we only have to show that $\mathrm{SNF}(Sp, X)$ is empty. Let us assume that $\phi$ satisfies $nf_1$ and $nf_2$ and show that $\phi$ does not satisfy $nf_3$. Because of $nf_1$, $\phi$ is $\alpha[t_y/y \mid y \in Var(\alpha)]$ for some $\alpha \in Ax$ and some $t_y \in T_\Sigma(X)$ s.t. $Fr \models_m D(t_y)$; $PSp$ being a positive conditional specification, all the premises of $\alpha$ may be assumed to be given as existential equalities. Thus $Fr \models_m \delta$ implies $A \models_V \delta$ for all models $A$, all valuations $V: X \to A$ and all $\delta \in \mathrm{prem}(\phi)$, by definition of $Fr$. Analogously $Fr \models_m D(t_y)$ implies $A \models_V D(t_y)$ and hence $V': Var(\alpha) \to A$, defined by $V'(y) = t_y^{A,V}$, is a valuation; moreover, $A$ being a model of $Sp$, $A \models_{V'} \alpha$ and hence $A \models_V \phi$. Therefore from $A \models_V \phi$ and $A \models_V \delta$ for all $\delta \in \mathrm{prem}(\phi)$, we get $A \models_V \mathrm{cons}(\phi)$ for all models $A$ and all valuations $V: X \to A$, so that $Fr \models_m \mathrm{cons}(\phi)$, i.e. $nf_3$ does not hold. □

Again using condition 5 of Theorem 2.7, we get the following basic undecidability result.

**Theorem 2.9.** *The existence of free objects is not decidable, even for finitary conditional specifications.*

**Proof.** We show that for every Thue system $E$ over an alphabet $A$ and every pair of nonempty strings $u$ and $w$ over $A$ there exists a specification $Sp_{E,u,w}$ s.t. $\mathrm{SNF}(Sp_{E,u,w}, \emptyset)$ is empty iff $u = w$ follows from $E$.

Therefore, since the set $\{(E, u, w) \mid E \vdash u = w\}$ is not decidable (see e.g. [6]) and the emptiness of $\mathrm{SNF}(Sp_{E,u,w}, \emptyset)$ is equivalent to the existence of an initial model for $Sp_{E,u,w}$, the existence of the initial model for the class of the conditional specifications $Sp_{E,u,w}$ is not decidable either. Thus we only have to exhibit such a conditional specification $Sp_{E,u,w}$.

It is well known that every Thue system $E$ over an alphabet $A$ may be represented by the total equational one-sorted specification

$$Sp_{E,A} = (\Sigma_A, E \cup \{\cdot(\cdot(x, y), z) = \cdot(x, \cdot(y, z))\}),$$

where $\Sigma_A$ consists of just one sort $s$, of a constant symbol $\underline{a}$ for each $a \in A$ and of a binary symbol $\cdot$ which represents the concatenation, in the sense that for all nonempty strings $u$ and $w$ over $A$ the equality $u = w$ follows from $E$ iff it holds in all models of $Sp_{E,A}$.

Then for each Thue system $E$ over $A$ and all nonempty strings $u$ and $w$ over $A$ let $Sp_{E,u,w}$ be the specification having the signature

$$\Sigma_A \cup (\{s'\}, \{b, b' \colon \to s'\})$$

and the axioms

$$E \cup \{\cdot(\cdot(x, y), z) = \cdot(x, \cdot(y, z)\} \cup \{D(\underline{a}) \mid a \in A\}$$
$$\{D(\cdot(x, y)), u = w \supset D(b), b = b' \supset D(b)\}.$$

It is easy to check that $\mathrm{SNF}(Sp_{E,u,w}, \emptyset)$ is empty iff $u = w$ follows from $E$. □

Finally we conclude the section showing that, w.r.t. model classes, partial conditional specifications are more expressive than the (total and) positive conditional ones, even if the attention is restricted to those specifications admitting initial models.

Note that there exist classes $PMod(Sp)$ admitting free objects for all families $X$ of variables which are not definable by only positive conditional formulas, as the following example shows.

**spec** $Sp_3 =$
  **sorts** $s$
  **opns**
    $a, b, c, d \colon \to s$
**axioms**
  $\alpha \quad a = b \supset c = d$

Then for all families $X$ of variables there exists a free object $(Fr, m)$ for $Sp_3$, defined by:

**Algebra** $Fr =$
  $s^{Fr} = X$
  $a^{Fr}, b^{Fr}, c^{Fr}, d^{Fr}$ are undefined
  $m(x) = x$

In order to show that $PMod(Sp_3)$ cannot be the model class of a positive conditional specification it is sufficient to show that it is not closed under nonempty products, since the model class of a positive conditional specification is a quasi-variety (see e.g.

[19]). Let $A$ and $B$ be the models of $Sp_3$ defined by:

**Algebra** $A =$
$s^A = \{1,2\}$
$a^A = 1, b^A$ undefined
$c^A = 2, d^A$ undefined

**Algebra** $B =$
$s^B = s^A$
$a^B$ undefined, $b^B = 1$
$c^B = 2$, $d^B$ undefined

Both are models of $Sp_3$; but their product is the algebra $C$ defined by:

**Algebra** $C =$
$s^C = \{1,2\} \times \{1,2\}$ $a^C, b^C, d^C$ undefined, $c^C = (2,2)$

that is not a model of $Sp_3$. $\square$

Both in the total conditional and in the partial positive conditional cases the closure under nonempty products of the model class guarantees that for any sort either the corresponding carrier is a singleton set in all models or there are models having this carrier larger than an arbitrary cardinality. Indeed assume that there is a model $A$ s.t. $s^A$ has cardinality at least 2; then for any (possibly infinite) set $I$ the product $\prod_I A$ is a model, because the model class is closed under nonempty products, and $s^{\prod_I A}$ has cardinality at least $2^I$. The lack of closure under nonempty products for the partial conditional case makes this property false; more precisely, denoting by $|X|$ the cardinality of any set $X$, for any $n \in \mathbb{N}$ there exists a conditional specification $Sp^n = (\Sigma^n, Ax^n)$ s.t.

- $|s^A| \leqslant n$ for all $A \in PMod(Sp^n)$ and all $s \in S$;
- there exists $A \in PMod(Sp^n)$ and $s \in S$ s.t. $|s^A| = n$.

Consider the following example.

**spec** $Sp_4 =$
**sorts** $s_i$ for $i = 1, \ldots, n$
**opns**
$a_{i,j}\colon \to s_i$ for $i = 1, \ldots, n$, $j = 1, \ldots, i$
$\xi_i : s_{i+1} \to s_i$ for $i = 1, \ldots, n-1$

**axioms**
$\alpha$ $\quad x = a_{1,1}$
$\beta_{i,j}$ $\quad D(a_{i,j})$ for $i = 1, \ldots, n$, $j = 1, \ldots, i$
$\gamma_i$ $\quad \xi_i(x) = \xi_i(y) \supset x = y$ for $i = 1, \ldots, n-1$

Let us assume that $A$ is a model of $Sp_4$ and inductively show that $|s_i^A| \leqslant i$ for $i = 1, \ldots, n$.

- $s_1^A$ is the singleton set $\{a_{1,1}^A\}$, because of $\alpha$;
- let us assume that $|s_i^A| \leqslant i$; for any $a \in s_{i+1}^A$ either $\xi_i^A(a) \in s_i^A$ or $\xi_i^A(a)$ is undefined and hence there are $|s_i^A| + 1$ possibilities to define $\xi_i^A(a)$ and hence, $\xi_i^A$ being an injective partial function because of $\gamma_i$, $|s_{i+1}^A| \leqslant |s_i^A| + 1 \leqslant i + 1$.

Moreover it is easy to check that $I$ is a model of $Sp_4$, where $I$ is defined by:

**Algebra** $I =$
$s_i^I = \{1, \ldots, i\}$
$a_{i,j}^I = j$ for all $i = 1, \ldots, n$ and all $j = 1, \ldots, i$
$\xi_i^I(j) = j$ for $j = 1, \ldots, i$ and $\xi_i^I(i + 1)$ is undefined for all $i = 1, \ldots, n - 1$

Note that in the total frame, from $|s_1^A| = 1$ and by the injectivity of $\xi_i^A$ we would get $|s_2^A| = 1$ and so, inductively, $|s_i^A| = 1$ for all $i$. In the partial positive frame, changing $\gamma_i$ in $\xi_i(x) =_e \xi_i(y) \supset x = y$, we allow many different elements to have undefined image along $\xi_i$ and hence there are models having the carriers of sort $s_i$ of arbitrary cardinality for all $i = 1, \ldots, n$. $\square$

## 3. Free objects and logical deduction

Here we discuss the relationships between the existence of the free models and logical deduction.

The central result (Theorem 3.6) characterizes the existence of free models in terms of completeness w.r.t. existential equalities of generic conditional inference systems. However, we start by illustrating the delicate issue of soundness and compare our solution to other approaches. That solution is illustrated by an important example of inference system, the $UL$ system, which is shown to be sound (Theorem 3.3). The $UL$ system is introduced both for the sake of concreteness and also as a preliminary to a complete system to be introduced later. As an application of our results we get completeness results w.r.t. existential equalities (but not w.r.t. strong equalities) for systems handling positive conditional specifications.

In the sequel when referring to generic formulas and inference systems we consider them within an infinitary logic which extends first-order logic by admitting denumerable conjunctions (, disjunctions) and quantification over denumerable sets of variables (see e.g. [12]).

Note that, as usual, quantification is always implicit and universal, as it may be easily deduced from the definition of validity, i.e. every formula $\phi$ is a short notation for the formula $\{\forall x\colon s \mid x \in Var(\phi)_s\}_{s \in S}.\phi$. However this short notation may cause a subtle error whenever empty carriers are allowed, as the following example shows.

**spec** $Sp_5$ =
  **sorts** $s_1, s_2$
  **opns**
    $a, b: \to s_1$
    $f: s_2 \to s_1$

  **axioms**
    $\alpha_1$ $D(a)$
    $\alpha_2$ $D(b)$
    $\alpha_3$ $a = f(x)$
    $\alpha_4$ $f(x) = b$

Now if $L$ is a logical system for $Sp_5$, then we obviously have $L \vdash a = f(x)$ and $L \vdash f(x) = b$, while $L \vdash a = b$ is an unsound deduction; for example $T_\Sigma$ is a model of $Sp_5$ (actually it is initial) but $T_\Sigma \not\models a = b$. This may happen, since $T_\Sigma(X)_{|s_2} = \emptyset$. □

Indeed Huet noted that, in the framework of many-sorted algebras, the family

$$\mathscr{R} = \{(t, t') \mid t, t' \in T_\Sigma(X)_{|s}, A \models t = t'\}_{s \in S}$$

may fail to be a congruence; in the example $T_\Sigma \models a = f(x)$ and $T_\Sigma \models f(x) = b$, but since $T_{\Sigma|s_2} = \emptyset$ and hence there exists no valuation for $\{x\}$ in $T_\Sigma$, $T_\Sigma \not\models a = b$; so that $\mathscr{R}$ is not transitive. He suggested how to avoid unsound ground deductions in the case of total algebras by restricting signatures to those whose corresponding carriers either are guaranteed to be nonempty by the existence of ground terms of each sort, or are in a sense absolutely disconnected by the nonempty carriers (the rigorous notion is that of sensible signature, see e.g. [11]). This approach fails in the partial framework since a ground term may be undefined in an algebra and hence its existence does not guarantee that the corresponding carrier is not empty; thus we cannot guarantee that a carrier is not empty just by imposing some restrictions on the signature.

The same problem was also tackled by Goguen and Meseguer in [15] with a particular interest to logical deduction. They proposed a system working on equalities of the form $(\forall X)t = t'$, where $Var(t) \cup Var(t') \subseteq X$, which produces $(\forall X - \{x\})t = t'$, eliminating a variable $x$ from $X$, only if $x$ does not appear in $t = t'$ and can be instantiated by a ground term. In this framework in $Sp_5$ we have that from $(\forall \{x\}) f(x) = b$ and $(\forall \{x\}) a = f(x)$ we deduce $(\forall \{x\}) a = b$, which holds also in $T_\Sigma$, but we cannot deduce $a = b$, as $x$ cannot be instantiated by a ground term, $T_{\Sigma|s_2}$ being empty. A similar approach can be used in the partial framework, permitting the elimination of those variables that can be instantiated by ground terms whose definedness is provable. For another system of equational deduction handling the empty carrier problem see [14].

However we can handle the problem in a way which is more natural for the partial approach; indeed we have at hand the existential equalities $t =_e t$, or definedness assertions $D(t)$, and we may use them to replace in a formula the explicit indication of the variables to which the valuation refers. Thus the ([15])-like formula $(\forall X)\Delta \supset \varepsilon$ becomes in our framework $\{D(x) \mid x \in X\} \cup \Delta \supset \varepsilon$. Moreover, since $D(y)$ holds in all algebras for any $y$, the presence of $D(y)$ among the premises of a conditional formula has only the effect of possibly increasing the set of variables appearing in the axiom. Hence we can forget the explicit indication of the definedness of the variables appearing in $\Delta \supset \varepsilon$. This makes the partial deduction more concise.

In order to make the presentation more concrete and to prepare the way to a completeness result, we introduce, for the moment just as an example, a particular system. It is reminiscent of systems found in the literature (see, e.g. [7]), but takes care of the soundness problem in the way suggested by the above remarks. We denote this system by $UL$, for Usual Logic system.

**Notation.** Since we will often need formulas of the form

$$\{D(x) \mid x \in X\} \cup \Theta \supset \varepsilon,$$

let us introduce a short notation for these formulas. For all families $X = \{X_s\}_{s \in S}$ of variables $D(X)$ will denote the set of formulas $\{D(x) \mid x \in X_s, s \in S\}$. Moreover for every inference system $L$, every family of variables $X$ and every elementary formula $\varepsilon$, if $L \vdash D(X') \supset \varepsilon$ for some $X' \subseteq X$, then we write $L \vdash_X \varepsilon$.

**Definition 3.1.** The $UL$ inference system consists of the following axioms and inference rules, where we assume that as usual $\varepsilon \in \mathrm{EForm}(\Sigma, Var)$, $\Delta, \Delta_\gamma, \Gamma$ are countable subsets of $\mathrm{EForm}(\Sigma, Var)$, $x \in Var$ and $t, t', t'', t_i, t'_i \in T_\Sigma(Var)$.

1. *Definedness of variables*

$$D(x)$$

2. *Congruence*

(a) $t = t$

(b) $t = t' \supset t' = t$

(c) $t = t' \wedge t' = t'' \supset t = t''$

(d) $t_1 = t'_1 \wedge \cdots \wedge t_n = t'_n \supset op(t_1, \ldots, t_n) = op(t'_1, \ldots, t'_n)$

3. *Strictness*

$$D(op(t_1, \ldots, t_n)) \supset D(t_i)$$

4. *Definedness and equality*

$$D(t) \wedge t = t' \supset D(t')$$

5. *Modus ponens*

$$\frac{\Delta \cup \Gamma \supset \varepsilon,\ \{\Delta_\gamma \supset \gamma \mid \gamma \in \Gamma\}}{D(Var(\Gamma) - Var(\bigcup_{\gamma \in \Gamma} \Delta_\gamma \cup \Delta \supset \varepsilon)) \cup \Delta \cup (\bigcup_{\gamma \in \Gamma} \Delta_\gamma) \supset \varepsilon}$$

6. *Instantiation/abstraction*

$$\frac{\Delta \supset \varepsilon}{\{D(t_x) \mid x \in X_s,\ s \in S\} \cup \{\delta[t_x/x \mid x \in X_s,\ s \in S] \mid \delta \in \Delta\} \supset \varepsilon[t_x/x \mid x \in X_s, s \in S]}$$

where, for $x \in X_s$, $t_x \in T_\Sigma(Var)_{|s}$.

If $Sp = (\Sigma, Ax)$ is a conditional specification, then $UL(Sp)$ denotes the system obtained from $UL$ by adding $Ax$ as axioms.

**Remark.** It is worth noting that instantiation and abstraction are both handled by the above rule 6 to keep the system as economical as possible; instantiation corresponds to $X$ being the family of variables of the formula that we are instantiating and abstraction corresponds to $X$ being a family of variables which do not appear in the formula that we are abstracting and $t_x = x$ for all $x \in X$. Thus rule 6 may be replaced by the following $*$ and $\star$.

$*$ *Instantiation*

$$\frac{\Delta \supset \varepsilon}{\{D(t_x) \mid x \in X_s, s \in S\} \cup \{\delta[t_x/x \mid x \in X_s, s \in S] \mid \delta \in \Delta\} \supset \varepsilon[t_x/x \mid x \in X_s, s \in S]}$$

where $X \subseteq Var(\Delta \supset \varepsilon)$

$\star$ *Abstraction*

$$\frac{\Delta \supset \varepsilon}{D(X) \cup \Delta \supset \varepsilon}$$

Obviously both $*$ and $\star$ are a particular case of 6 and it is easy to check that any application of 6 may be replaced by an application of $\star$ to increase the number of variables and an application of $*$ to instantiate the variables:

$$\frac{\Delta \supset \varepsilon}{D(X) \cup \Delta \supset \varepsilon}$$

$$\frac{D(X) \cup \Delta \supset \varepsilon}{\Gamma_{\mathrm{Def}} \cup \{\delta[t_x/x \mid x \in X_s, s \in S] \mid \delta \in \Delta\} \supset \varepsilon[t_x/x \mid x \in X_s, s \in S]}$$

where $\Gamma_{\mathrm{Def}} = \{D(t_x) \mid x \in X_s,\ s \in S\}$.

The focus of algebraic logic deduction is on *equational* deduction, because an inference system complete w.r.t. the (existential) equations gives the (initial) free model, if any. Since equations with explicit quantification $\forall(X \cup Var(\varepsilon)).\varepsilon$ are equivalent in our approach to formulas of the form $D(X) \supset \varepsilon$, we give notions of soundness and completeness also dealing with such particular conditional formulas; our notions subsume the usual ones dealing only with equalities.

**Definition 3.2.** Let $L$ be an inference system with conditional formulas over $\Sigma$ and $Sp$ be a conditional specification over $\Sigma$.

- $L$ *respects congruence* if for all families $X$ of variables the family $\equiv^L(X) = \{\equiv^L(X)_s\}_{s \in S}$, where

  $$\equiv^L(X)_s = \{(t,t') \mid t,t' \in T_\Sigma(X)_{|s}, L \vdash_X D(t), L \vdash_X t = t'\}$$

  is a congruence over $T_\Sigma(X)$ s.t.

  $$Dom(\equiv^L(X)_s) = \{t \mid t \in T_\Sigma(X)_{|s}, L \vdash_X D(t)\} \supseteq X.$$

- $L$ is *sound* w.r.t. $Sp$ if for any formula $\phi$, $L \vdash \phi$ implies $M \models \phi$ for all $M \in PMod(Sp)$.
- $L$ is a (*conditional*) *system* for $Sp$ if it respects congruences and is sound w.r.t. $Sp$.

Let $\mathrm{EEq}(L, X)$ denote the following set

$$\{D(t) \mid t \in T_\Sigma(X)_{|s}\} \cup \{t = t' \mid t,t' \in T_\Sigma(X)_{|s}, L \vdash_X D(t) \textit{ or } L \vdash_X D(t')\}.$$

- A system $L$ for $Sp$ is *existentially equationally complete for $X$ and $Sp$*, in the sequel simply called eeq-complete, iff for any $\varepsilon \in \mathrm{EEq}(L, X)$ if $M \models D(X) \supset \varepsilon$ for all $M \in PMod(Sp)$, then there exists $X' \subseteq X$ s.t. $L \vdash D(X') \supset \varepsilon$.
- A system $L$ for $Sp$ is *strongly equationally complete for $X$ and $Sp$*, in the sequel simply called seq-complete, iff for any elementary formula $\varepsilon$ over $\Sigma$ and $X$ if $M \models D(X) \supset \varepsilon$ for all $M \in PMod(Sp)$, then there exists $X' \subseteq X$ s.t. $L \vdash D(X') \supset \varepsilon$.

**Remark.** It is worth noting that the easier formulation of completeness ...if $M \models D(X) \supset \varepsilon$ for all $M \in PMod(Sp)$, then $L \vdash D(X) \supset \varepsilon$...
is too restrictive. Indeed, although most systems have a rule of abstraction which allows to deduce $L \vdash D(X') \cup \{x\} \supset \varepsilon$ from $L \vdash D(X') \supset \varepsilon$, in general if $X$ is an infinite set, then $L \vdash D(X') \supset \varepsilon$ does not imply $L \vdash D(X) \supset \varepsilon$ and in particular this happens for any finitary system, as the one that will be presented in Section 4.

**Theorem 3.3.** *For all conditional specifications $Sp$, $UL(Sp)$ is a system for $Sp$, i.e. it respects congruences and is sound w.r.t. $Sp$.*

**Proof.** Rules 1–5 ensure that $UL(Sp)$ respects congruence; thus we only have to show that it is sound. We do this by induction over the rules of $UL(Sp)$. It is obvious that the rules 1–4 are sound, by definition of validity for the definedness predicate and the equality; thus we only consider rules 5 and 6.

Assume that the hypotheses of rule 5 are satisfied, i.e. that $UL(Sp) \vdash \phi$, where $\phi$ is $\Delta \cup \Gamma \supset \varepsilon$, and that $UL(Sp) \vdash \phi_\gamma$ for all $\gamma \in \Gamma$, where $\phi_\gamma$ is $\Delta_\gamma \supset \gamma$. Then $UL(Sp) \vdash \phi'$, where $\phi'$ is

$$D\left(Var(\Gamma) - Var\left(\Delta \cup \bigcup_{\gamma \in \Gamma} \Delta_\gamma \supset \varepsilon\right)\right) \cup \Delta \cup \left(\bigcup_{\gamma \in \Gamma} \Delta_\gamma\right) \supset \varepsilon,$$

and we have to show that $A \models \phi$ and $A \models \phi_\gamma$ for all $\gamma \in \Gamma$ implies $A \models \phi'$ for all $A \in PMod(Sp)$. Let $V$ be a valuation for $Var(\phi')$ in $A \in PMod(Sp)$ s.t. $A \models_V \delta$ for all $\delta \in \mathrm{prem}(\phi')$; we show that $A \models_V \mathrm{cons}(\phi')$.

First of all note that $Var(\phi')$ is the same set as $Var(\phi) \cup \bigcup_{\gamma \in \Gamma} Var(\phi_\gamma)$, so that $V$ is also a valuation for $Var(\phi)$ and for $Var(\phi_\gamma)$ in $A$. Moreover for all $\gamma \in \Gamma$ we have that $\mathrm{prem}(\phi_\gamma) \subseteq \mathrm{prem}(\phi')$ and hence, because of $A \models_V \delta$ for all $\delta \in \mathrm{prem}(\phi')$, $A \models_V \delta$ for all $\delta \in \mathrm{prem}(\phi_\gamma)$; thus, since $A \models \phi_\gamma$, $A \models_V \gamma$ for all $\gamma \in \Gamma$. Therefore, since $\Delta \subseteq \mathrm{prem}(\phi')$, $A \models_V \delta$ for all $\delta \in \mathrm{prem}(\phi) = \Gamma \cup \Delta$, so that $A \models_V \mathrm{cons}(\phi)$, because $A \models_V \phi$. Finally $\mathrm{cons}(\phi) = \mathrm{cons}(\phi')$ and hence $A \models_V \phi'$.

Similarly we proceed in the case of rule 6. Assume that $UL(Sp) \vdash \phi$, where $\phi$ is $\Delta \supset \varepsilon$; then $UL(Sp) \vdash \phi'$, where $\phi'$ is the formula

$$\{D(t_x) \mid x \in X_s,\ s \in S\} \cup \{\delta[t_x/x \mid x \in X_s,\ s \in S] \mid \delta \in \Delta\} \supset \varepsilon[t_x/x \mid x \in X_s,\ s \in S],$$

and we have to show that $A \models \phi$ implies $A \models \phi'$ for all $A \in PMod(Sp)$.

Then let $V$ be a valuation for $Var(\phi')$ in $A \in PMod(Sp)$ s.t. $A \models_V \delta$ for all $\delta \in \mathrm{prem}(\phi')$ and show that $A \models_V \mathrm{cons}(\phi')$. Since $\{D(t_x) \mid x \in X_s, s \in S\} \subseteq \mathrm{prem}(\phi')$, $A \models_V D(t_x)$ for all $x \in X$; thus $V' : Var(\phi) \to A$ defined by $V'(x) = t_x^{A,V}$ if $x \in X$, otherwise $V'(x) = V(x)$ is a valuation for $Var(\phi)$ in $A$. Moreover, by definition of $V'$, for all elementary formulas $\delta$ on $Var(\phi)$ we have that $A \models_{V'} \delta$ iff $A \models_V \delta[t_x/x \mid x \in X_s,\ s \in S]$. Thus $A \models_{V'} \delta$ for all $\delta \in \mathrm{prem}(\phi)$ so that, as $A \models \phi$, $A \models_{V'} \mathrm{cons}(\phi)$, i.e. $A \models_{V'} \varepsilon$ and hence $A \models_V \varepsilon[t_x/x \mid x \in X_s,\ s \in S]$. $\square$

We now characterize completeness in terms of free models.

**Notation.** Let $Sp$ be a conditional specification, $X$ be a family of variables and $L$ be a system for $Sp$. In the sequel $\mathsf{Fr}(L,X)$ stands for $T_\Sigma(X)/\equiv^L(X)$ and $\mathsf{m}(L,X) : X \to \mathsf{Fr}(L,X)$ is the valuation defined by $\mathsf{m}(L,X)(x) = [x]$.

Note that $\mathsf{Fr}(L,X)$ is well defined because $\equiv^L(X)$ is a congruence and that $\mathsf{m}(L,X)$ is really a valuation, i.e. a total function, because $X \subseteq Dom(\equiv^L(X))$.

Any formula in $\mathrm{EEq}(L,X)$ plays the role of a (quantified) existential equality; this justifies calling "existentially equational completeness" the completeness w.r.t. $\mathrm{EEq}(L,X)$. Thus an eeq-complete system deduces all existential equalities holding in all models and hence $\mathsf{Fr}(L,X)$ is exactly $Fr(Sp,X)$.

**Proposition 3.4.** *For all conditional systems $L$ for $Sp$ the system $L$ is eeq-complete for $X$ and $Sp$ iff* $\mathsf{Fr}(L,X)$ *coincides with $Fr(Sp,X)$.*

**Proof.** Let us denote $K(Sp,X)$ by $K$ and $\equiv^L(X)$ by $\equiv$; by definition $\mathsf{Fr}(L,X)$ coincides with $\mathsf{Fr}(Sp,X)$ iff $\equiv$ and $K$ are the same.

Because of the soundness of $L$, $\equiv\ \subseteq K$. Indeed if $(t,t') \in\ \equiv$, then, by definition of $\equiv$, there exist $X', X'' \subseteq X$ s.t. $L \vdash D(X') \supset D(t)$ and $L \vdash D(X'') \supset t = t'$. Thus, $L$ being sound, for all models $A$ and all valuations $V$ for $X$ in $A$, $A \models_V D(X') \supset D(t)$ and $A \models_V D(X'') \supset t = t'$; moreover, by definition of valuation, $A \models_V D(x)$ for all $x \in X$ and hence $A \models_V D(t)$, $A \models_V t = t'$, i.e. $(t, t') \in K^{A,V}(X)$. Therefore $(t, t') \in K$.

So we have to show that $K \subseteq\ \equiv$ iff $L$ is eeq-complete.

($\Rightarrow$) Let us denote formulas in $\mathrm{EEq}(L, X)$ by existential equalities. We assume that $t =_e t' \in \mathrm{EEq}(L, X)$ and $L \not\vdash D(X') \supset t =_e t'$ for all $X' \subseteq X$ and show that there exists a model $A \in PMod(Sp)$ and a valuation $V: X \to A$ s.t. $A \not\models_V t =_e t'$. Since $L \not\vdash D(X') \supset t =_e t'$ for any $X' \subseteq X$, $(t, t') \notin\ \equiv$ and hence, $\equiv$ being equal to $K$, $(t, t') \notin K$. Thus there exists a model $A \in PMod(Sp)$ and a valuation $V: X \to A$ s.t. $A \not\models_V t =_e t'$.

($\Leftarrow$) If $(t, t') \in K$, then for all models $A$ and all valuations $V: X \to A$, $(t, t') \in K^{A,V}(X)$, i.e. $A \models_V D(t)$, $A \models_V t = t'$; hence $A \models_V D(X) \supset D(t)$ and $A \models_V D(X) \supset t = t'$. Thus, as $L$ is eeq-complete, $L \vdash_X D(t)$ and $L \vdash_X t = t'$, i.e. $(t, t') \in\ \equiv$. $\square$

Note that, as the following example shows, in general a conditional system is not eeq-complete for conditional specifications, even under the more restrictive hypotheses that there exists a free object for every family of variables; moreover the free object may be different from $(\mathsf{Fr}(L, X), \mathsf{m}(L, X))$.

**spec** $Sp_6$ =
  **sorts** $s$
  **opns**
    $a : \to s$
    $f, g : s \to s$

  **axioms**
    $\alpha_1$ $f(x) = g(x) \supset D(f(a))$
    $\alpha_2$ $D(f(x)) \supset f(x) = g(x)$
    $\alpha_3$ $D(g(x)) \supset f(x) = g(x)$
    $\alpha_4$ $D(a)$

Because of $\alpha_2$ and $\alpha_3$ we have that $f(x) = g(x)$ holds in all models of $Sp_6$ and hence, instantiating $\alpha_1$ for $x = a$, that is defined by $\alpha_4$, $D(f(a))$ holds too, while, for example, $UL(Sp_6) \not\vdash D(f(a))$ and hence $UL(Sp_6)$ is not eeq-complete for $\emptyset$ and $Sp_6$. Moreover for every family $X$ of variables $Fr(Sp_6, X)$, defined explicitly below, is a model and hence, because of Theorem 2.7, it is the free object for $X$ in $PMod(Sp_6)$.

**Algebra** $Fr(SP_6, X)$ =
  $s^{Fr(SP_6, X)} = X \cup \{1, 2\}$
  $a^{Fr(SP_6, X)} = 1$
  $f^{Fr(SP_6, X)} = \phi = g^{Fr(SP_6, X)}$

  – where $\phi$ is defined only on 1 and $\phi(1) = 2$.

Thus in general the existence of a free object does not imply that $\mathsf{Fr}(L,X)$ is a model; however if $\mathsf{Fr}(L,X)$ is a model, then it is also the free object for $X$ in $PMod(Sp)$, because $L$ is sound, as the following proposition shows.

**Proposition 3.5.** *For all families $X$ of variables and all systems $L$ for $Sp$ the algebra $\mathsf{Fr}(L,X)$ is a model of $Sp$ iff $(\mathsf{Fr}(L,X), \mathsf{m}(L,X))$ is free for $X$ in $PMod(Sp)$.*

**Proof.** Let $\mathsf{Fr}$ denote $\mathsf{Fr}(L,X)$ and $\mathsf{m}$ denote $\mathsf{m}(L,X)$.

($\Rightarrow$) Because of Theorem 2.7, we only have to show that $\equiv^L(X) = K(Sp,X)$. As we have seen in Proposition 3.4, $\equiv^L(X) \subseteq K(Sp,X)$, because of soundness of $L$. If $\mathsf{Fr}$ is a model, then $K(Sp,X) \subseteq K^{\mathsf{Fr},\mathsf{m}} = \equiv^L(X)$; thus $K(Sp,X) = \equiv^L(X)$.

($\Leftarrow$) Obvious. $\square$

It is now convenient to give a notion of naughty formula related to a system, since it allows us to connect the existence of a free model with logical inference systems.

**Notation.** Let $L$ be a system for a conditional specification $Sp$ and $X$ be a family of variables. We denote the set $\mathrm{NF}(Sp, \equiv^L(X))$ by $\mathrm{NF}(L,X)$.

Putting together Propositions 3.4, and 3.5 and Lemma 2.6 we get a theorem summarizing the relationships between systems and free models.

**Theorem 3.6.** *Let $Sp$ be a conditional specification, $X$ be a family of variables, $\mathsf{Fr}$ denote $\mathsf{Fr}(L,X)$ and $\mathsf{m}$ denote $\mathsf{m}(L,X)$. For every system $L$ for $Sp$ the following conditions are equivalent:*

1. *the set $\mathrm{NF}(L,X)$ is empty;*
2. *the algebra $\mathsf{Fr}$ is a model of $Sp$;*
3. *the pair $(\mathsf{Fr},\mathsf{m})$ is free for $X$ in $PMod(Sp)$.*

*If (one of) the above conditions hold, then $L$ is eeq-complete for $X$ and $Sp$ and $\mathsf{Fr}(L,X) = Fr(Sp,X)$.*

*If $L$ is a eeq-complete then each one of the above conditions is equivalent to*

4. *there exists a free object for $X$ in $PMod(Sp)$;*
5. *there exists a free object for $X$ in $PGen(Sp,X)$.*

**Proof.**

($1 \Leftrightarrow 2$) By Lemma 2.6.

($2 \Leftrightarrow 3$) By Proposition 3.5.

Let us assume that one among conditions 1, 2 and 3 holds, and show that $L$ is eeq-complete and that $\mathsf{Fr}(L,X) = Fr(Sp,X)$. Let $t =_e t'$ be a formula of $\mathrm{EEq}(L,X)$ s.t. $L \not\vdash D(X') \supset t =_e t'$ for any $X' \subseteq X$. Then, by definition of $\mathsf{Fr}$ and $\mathsf{m}$, $\mathsf{Fr} \not\models_{\mathsf{m}} t =_e t'$.

Hence, Fr being a model of $Sp$, $L$ is eeq-complete. Thus, because of Proposition 3.4, $\mathsf{Fr}(L, X) = Fr(Sp, X)$.

Assume now that $L$ is eeq-complete.

(4 $\Leftrightarrow$ 5) Because of Theorem 2.7

(4 $\Leftrightarrow$ 2) Since $L$ is eeq-complete, $\mathsf{Fr} = Fr(Sp, X)$ because of Proposition 3.4 and hence we conclude by Theorem 2.7. $\square$

In the cases of positive conditional (see e.g. [8]) and total conditional (see e.g. [15]) specifications, quite natural conditions on the inference system can be found that guarantee both the eeq-completeness of the system and the existence of a free model. We first state a preliminary result.

**Proposition 3.7.** *Let $PSp = (\Sigma, Ax)$ be a positive conditional specification, $X$ be a family of variables and $L$ be a conditional system for $Sp$ s.t.*
*Axioms. $L \vdash \alpha$ for all $\alpha \in Ax$.*
*Substitution. For all conditional formulas $\Delta \supset \eta$, all families $X \subseteq Var(\Delta \supset \eta)$, $Z_x$ of variables s.t. $L \vdash D(Z_x) \supset D(t_x)$ for all $x \in X$, $L \vdash \Delta \supset \eta$ implies*

$$L \vdash D\left(\bigcup_{x \in X} Z_x\right) \cup \{\delta[t_x/x \mid x \in X] \mid \delta \in \Delta\} \supset \eta[t_x/x \mid x \in X].$$

*Modus ponens. For any countable set of elementary formulas $\Theta$, $\Gamma$, $\Theta_\gamma$ and any elementary formula $\varepsilon$ $L \vdash \Theta \cup \Gamma \supset \varepsilon$ and $L \vdash \Theta_\gamma \supset \gamma$ for all $\gamma \in \Gamma$ implies*

$$L \vdash D(Y) \cup \Theta \cup \left(\bigcup_{\gamma \in \Gamma} \Theta_\gamma\right) \supset \varepsilon$$

*for some $Y \subseteq \bigcup_{\gamma \in \Gamma} Var(\gamma)$.*
*Then the following holds.*

1. *For every $\alpha \in Ax$, any instantiation of $\alpha$ does not belong to* $\mathrm{NF}(L, X)$.
2. ($\mathsf{Fr}(L, X)$, $\mathsf{m}(L, X)$) *is free for $X$ in $PMod(PSp)$;*
3. *$L$ is eeq-complete for $Sp$ and $X$.*

**Proof.**

1. Assume that $\phi$ is an instantiation of $\alpha$ satisfying $nf_1$ and $nf_2$ (cf. Definition 2.5) and show that $\phi$ does not satisfy $nf_3$. Since $\phi$ is an instantiation of $\alpha$ satisfying $nf_1$, $\phi = \alpha[t_y/y \mid y \in Var(\alpha)]$ for some $t_y \in T_\Sigma(X)$ s.t. there exists $X_y$ s.t. $L \vdash D(X_y) \supset D(t_y)$ and hence, because of the condition on deducibility of the axioms and instantiation for conditional systems, $L \vdash D(\bigcup_{y \in Var(\alpha)} X_y) \cup \mathrm{prem}(\phi) \supset \mathrm{cons}(\phi)$. Since $\alpha$ is a positive conditional axiom, we can assume that all premises of $\phi$ are existential equalities; thus, by definition of $\mathsf{Fr}(L, X)$, condition $nf_2$, i.e. $\mathsf{Fr}(L, X) \models_{\mathsf{m}(L,X)} \delta$ for all $\delta \in \mathrm{prem}(\phi)$, implies that there exists $X_\delta \subseteq X$ s.t. $L \vdash D(X_\delta) \supset \delta$ for every $\delta \in \mathrm{prem}(\phi)$.

Therefore by $L \vdash D(\bigcup_{y \in Var(x)} X_y) \cup \text{prem}(\phi) \supset \text{cons}(\phi)$ and $L \vdash D(X_\delta) \supset \delta$ for all $\delta \in \text{prem}(\phi)$, because of condition on modus ponens,

$$L \vdash D\left(\bigcup_{y \in Var(x)} X_y \cup \bigcup_{\delta \in \text{prem}(\phi)} X_\delta \cup X'\right) \supset \text{cons}(\phi)$$

for some $X' \subseteq Var(\text{prem}(\phi)) \subseteq X$, i.e. condition $nf_3$ does not hold.

2. By the first point, $\text{NF}(L, X)$ is empty; thus, by Theorem 3.6, we have that $(\mathsf{Fr}(L, X), \mathsf{m}(L, X))$ is free for $X$ in $PMod(PSp)$.

3. By the second point and Theorem 3.6, $L$ is eeq-complete. □

Since total conditional specifications are a special case of (partial) positive conditional ones, this result applies to that case too and hence we get also the freeness result for total conditional specifications of [15].

The conditions on substitution, modus ponens and axiom deducibility of Proposition 3.7 are satisfied by the $UL(Sp)$ system, which implies the following.

**Corollary 3.8.** *For any positive conditional specification PSp, the system* $UL(PSp)$ *is eeq-complete for Sp and X and* $(\mathsf{Fr}(UL(PSp), X), \mathsf{m}(UL(PSp), X))$ *is free for X in* $PMod(PSp)$ *for every X.*

**Proof.** Since the axioms of $PSp$ belong to $UL(PSp)$, the condition on the axioms is satisfied; moreover, because of rules 5 and 6, $UL(PSp)$ satisfies the condition on substitution and, because of rule 5, the condition on modus ponens. Therefore Proposition 3.7 applies. □

In general, however, a conditional system is not seq-complete for positive conditional specifications, as the following example shows.

**spec** $Sp_7$ =
  **sorts** $s$
  **opns**
    $a : \to s$
    $f, g : s \to s$
  **axioms**
    $\alpha_1$ $D(f(x)) \supset f(x) = g(x)$
    $\alpha_2$ $D(g(x)) \supset f(x) = g(x)$

Since one among $D(f(x))$, $D(g(x))$ and $f(x) = g(x)$ holds by definition of strong equality, from $\alpha_1$ and $\alpha_2$ we have that $f(x) = g(x)$ holds in all models of $Sp_7$, while, for example, $UL(Sp_7) \not\vdash f(x) = g(x)$ and hence $UL(Sp_7)$ is not seq-complete for $\emptyset$ and $Sp_7$. □

## 4. Complete conditional systems

In this section we present and prove completeness of systems w.r.t. strong equalities. These systems are obtained from the $UL$ system of the previous section just by adding one crucial (and subtle) elimination rule.

In the first subsection two systems are presented, one for dealing with equalities with variables ($CL$) and the other for dealing with ground equalities.

Their completeness, which is by far the most difficult result of the paper (Theorems 4.5 and 4.6) is then proved in the second subsection.

Finally a simplified system for the finitary case (i.e. only finite sets of premises in the axioms), where the elimination rule has a very simple and intuitive form, is given in the third subsection, where its completeness is nontrivially derived from the completeness of the system for the infinitary case (Theorem 4.19).

### *4.1. Introducing the systems*

If we consider a positive conditional specification $PSp$ and a system $L$ for $PSp$, then we have seen that $(\mathsf{Fr}(L,X),\mathsf{m}(L,X))$ is free for $X$ in $PMod(PSp)$. However, if we consider a conditional specification $Sp$, even if there exists a free model for $X$ in $PMod(Sp)$, a generic system $L$ for $Sp$ may be too poor for $(\mathsf{Fr}(L,X),\mathsf{m}(L,X))$ to be the free model for $X$ in $PMod(Sp)$; for instance, even if there exists a free model in $PMod(Sp)$, $\mathsf{Fr}(UL(Sp),X)$ may be not a model of $Sp$, as we have seen in Section 3 in the case of the specification $Sp_6$. Therefore we devote this section to exhibiting an eeq-complete conditional system.

A most interesting feature of this system is that it is obtained by adding just one new rule to the system $UL(Sp)$ of the previous section. It is convenient to introduce that rule with a preliminary discussion.

Let us consider a more significant example of specification $Sp$ having free models for all families of variables but s.t. $\mathsf{Fr}(UL(Sp),\emptyset)$ is not a model of $Sp$.

**spec** $Sp_8$ =
  **sorts** $s_1, s_2$
  **opns**
    $a, b :\rightarrow s_1$
    $e :\rightarrow s_2$
  **axioms**
    $\alpha_1 \quad D(a) \supset D(e)$
    $\alpha_2 \quad a = b \supset D(e)$
    $\alpha_3 \quad D(b) \supset D(e)$

In the models of $Sp_8$, $D(a)$ or $D(b)$ or $a = b$ holds, by definition of strong equality; so that because of $\alpha_1$, $\alpha_2$ and $\alpha_3$ we have that also $D(e)$ has to hold in the models of $Sp_8$, while $UL(Sp_8) \not\vdash D(e)$. Moreover for all families $X$ of variables the free object $(Fr(X), m)$ for $X$ in $PMod(Sp_8)$ is defined by:

**Algebra** $Fr(X) =$
$s_1^{Fr(X)} = X_{s_1}$
$s_2^{Fr(X)} = X_{s_2} \cup \{\bullet\}$
$a^{Fr(X)}, b^{Fr(X)}$ are undefined
$e^{Fr(X)} = \bullet$
$m(x) = x$

The specification $Sp_8$ suggests that, to make $UL(Sp)$ complete, we have to add a rule $\star$ having basically the form

$$\frac{(\Delta_1 \cup \{D(t)\}) \supset \varepsilon,\ (\Delta_2 \cup \{D(t')\}) \supset \varepsilon,\ (\Delta_3 \cup \{t = t'\}) \supset \varepsilon}{(\Delta_1 \cup \Delta_2 \cup \Delta_3) \supset \varepsilon}$$

where $t$ and $t'$ are ground terms.

If $t$ and $t'$ are not ground, we have obviously to generalize the above rule by keeping track of the variables in $t$ and $t'$ in the way introduced in Section 3. Moreover, since we are working within infinitary logic, we have to generalize the above rule $\star$ to eliminate an infinite number of premises in one step.

To present some intuition about the required generalization, let us first consider a finitary case where there are more than one strong equalities to eliminate (even though every finitary case can be handled by a finite number of applications of $\star$; see Section 4.3 below).

**spec** $Sp_9 =$
**sorts** $s_1, s_2$
**opns**
$a, b, c, d : \to s_1$
$e : \to s_2$
**axioms**
$\alpha_1$ $D(a) \wedge D(c) \supset D(e)$
$\alpha_2$ $a = b \wedge D(c) \supset D(e)$
$\alpha_3$ $D(b) \wedge D(c) \supset D(e)$
$\alpha_4$ $D(a) \wedge D(d) \supset D(e)$
$\alpha_5$ $a = b \wedge D(d) \supset D(e)$
$\alpha_6$ $D(b) \wedge D(d) \supset D(e)$
$\alpha_7$ $D(a) \wedge c = d \supset D(e)$
$\alpha_8$ $a = b \wedge c = d \supset D(e)$
$\alpha_9$ $D(b) \wedge c = d \supset D(e)$

In all models of $Sp_9$ at least one among $D(a)$ or $D(b)$ or $a = b$ holds, by definition of strong equality, and analogously at least one among $D(c)$ or $D(d)$ or $c = d$ holds.

Therefore in all models of $Sp_9$ the premises of at least one among $\alpha_1, \dots, \alpha_9$ hold and hence we conclude that $D(e)$ holds in the models of $Sp_9$.

Note that in all models of $Sp_9$ the premises of at least one axiom hold since $\{\text{prem}(\alpha_i\} \mid i = 1, \dots, 9\}$ is the set $\{D(a), D(b), a = b\} \times \{D(c), D(d), c = d\}$ and one among $\{D(a), D(b), a = b\}$ and one among $\{D(c), D(d), c = d\}$ has to hold. Then for a generic finitary case we deduce from a family $\{\phi_j \mid j = 1, \dots, n\}$ of conditional formulas an elementary formula $\varepsilon$ iff:

- $\text{cons}(\phi_j) = \varepsilon$ for all $j = 1, \dots, n$;
- $\{\text{prem}(\phi_j) \mid j = 1, \dots, n\}$ is the set

$$\{D(t_1), D(t'_1), t_1 = t'_1\} \times \cdots \times \{D(t_m), D(t'_m), t_m = t'_m\}$$

for some $t_i, t'_i$ and $i = 1, \dots, m$.

Indeed in all models $A$ one among $D(t_i), D(t'_i)$, $t_i = t'_i$ holds for all $i = 1, \dots, m$ and hence there exist $j \in \{1, \dots, n\}$ s.t. $A \models \delta$ for all $\delta \in \text{prem}(\phi_j)$.

The point is that we have a set $\{\phi_j \mid j = 1, \dots, n\}$ of conditional formulas with the same consequence and s.t. the premises of the $\phi_j$ cannot be simultaneously falsified, because any choice of one element in every $\text{prem}(\phi_j)$ is always true, containing the set $\{D(t), D(t'), t = t'\}$.

Let us summarize the above discussion by a formal statement.

**Definition 4.1.** For $Sp = (\Sigma, Ax)$, the inference system $CL(Sp)$ consists of the axioms and inference rules of $UL(Sp)$ and of the following inference rule:
*Elimination*

$$\frac{\{\Theta_j \cup \Gamma_j \supset \varepsilon \mid j \in J\}}{D(\bigcup_{j \in J} Var(\Gamma_j)) \cup (\bigcup_{j \in J} \Theta_j) \supset \varepsilon}$$

where

- $\varepsilon \in \text{EForm}(\Sigma, Var)$;
- $\Gamma = \{\Gamma_j \mid j \in J\}$ and $\Theta = \{\Theta_j \mid j \in J\}$ are families of countable subsets of $\text{EForm}(\Sigma, Var)$ indexed by a possibly noncountable $J$;
- $\Gamma$ satisfies the following *ininfluence condition*:
  for all $\Psi = \{\gamma_j \mid j \in J\}$ s.t. for all $j \in J$, $\gamma_j \in \Gamma_j$, there exist $t, t' \in T_\Sigma(Var)$ s.t. $D(t)$, $D(t')$, $t = t' \in \Psi$.
  We say that $\Psi$ is a *section* of $\Gamma$ and we denote by $\text{Sec}(\Gamma)$ the set of all sections of $\Gamma$; moreover under this condition we say that $\Gamma$ is *ininfluent*.

The intuition that $\Gamma = \{\Gamma_j \mid j \in J\}$ is ininfluent iff $\Gamma$ is, in some sense, $\{D(t_1), D(t'_1), t_1 = t'_1\} \times \cdots \times \{D(t_m), D(t'_m), t_m = t'_m\} \times \cdots$ is formalized by the following proposition.

**Proposition 4.2.** *Let $\Gamma = \{\Gamma_j \mid j \in J\}$ be any family of sets of elementary formulas. Then $\Gamma$ is ininfluent iff there exist a set $I$ s.t. for any section $\sigma$ of $\mathscr{R} = \{R(t_i, t'_i) \mid i \in I\}$, where $R(t,t') = \{D(t), D(t'), t = t'\}$, there exists $j \in J$ s.t. $\Gamma_j \subseteq \sigma$.*

*Moreover if $J$ and $\Gamma_j$ for all $j \in J$ are finite, then $I$ is finite too.*

**Proof.** ($\Rightarrow$) Let us consider the set $C$ of all possible pairs of terms appearing in $\Gamma$; then $C = \{(t_i, t'_i) \mid i \in I\}$, where $I$ is any set with the same cardinality as $C$; thus if $C$ is finite (in particular if $J$ and $\Gamma_j$ for all $j \in J$ are finite), then $I$ is finite, too.

Let us assume by contradiction that there exists a section $\sigma$ of $\mathscr{R} = \{R(t_i, t'_i) \mid i \in I\}$ s.t. $\Gamma_j \subseteq \sigma$ does not hold for any $j \in J$, i.e. there exists $\gamma_j \in \Gamma_j$ s.t. $\gamma_j \notin \sigma$ for any $j \in J$. Thus, by definition, $\Psi = \{\gamma_j \mid j \in J\}$ is a section for $\Gamma$ and $\Psi \cap \sigma = \emptyset$. Because of the ininfluence of $\Gamma$, there exists $n \in I$ s.t. $R(t_n, t'_n) \subseteq \Psi$. Then $\Psi \cap \sigma \neq \emptyset$, as $\sigma$ is a section of $\mathscr{R}$, contrary to the construction of $\Psi$.

($\Leftarrow$) If $\Gamma_j = \emptyset$ for some $j \in J$, then $\Gamma$ is obviously ininfluent; otherwise let $\Psi$ be a section of $\Gamma$ and assume by contradiction that there do not exist terms, $t, t'$ s.t. $R(t,t') \subseteq \Psi$. So in particular for any $i \in I$ there exists $\sigma_i \in R(t_i, t'_i)$ s.t. $\sigma_i \notin \Psi$ and hence $\Psi \cap \sigma = \emptyset$, where $\sigma$ is the section $\{\sigma_i \mid i \in I\}$ of $\mathscr{R}$. By hypothesis there exists $j \in J$ s.t. $\emptyset \neq \Gamma_j \subseteq \sigma$ and hence, $\Psi$ being a section of $\Gamma$, $\Psi \cap \sigma \neq \emptyset$, contrary to the construction of $\sigma$. $\square$

**Proposition 4.3.** *The inference system $CL(Sp)$ is a system for $Sp$, i.e. it respects congruences and is sound w.r.t. $Sp$.*

**Proof.** It is sufficient to show that the elimination rule is sound.

Let us assume that $A$ belongs to $PMod(Sp)$ and $A \models_V \theta$ for a suitable valuation $V$ for $Var(D(Var(\bigcup_{j \in J} \Gamma_j)) \cup (\bigcup_{j \in J} \Theta_j) \supset \varepsilon)$ and all $\theta \in (D(Var(\bigcup_{j \in J} \Gamma_j)) \cup (\bigcup_{j \in J} \Theta_j))$. We show that $A \models_V \varepsilon$. It is sufficient to prove that there exists $j \in J$ s.t. $A \models_V \theta$ for all $\theta \in \Theta_j \cup \Gamma_j$, because $A \models (\Theta_j \cup \Gamma_j) \supset \varepsilon$ for all $j \in J$ by inductive hypothesis, and $V$ is also a valuation for $Var(\Theta_j \cup \Gamma_j \supset \varepsilon)$ in $A$.

Since we have assumed that $A \models_V \theta$ for all $\theta \in \bigcup_{j \in J} \Theta_j$, we have that $A \models_V \theta$ for all $\theta \in \Theta_j$ and all $j \in J$; thus we only have to show that there exists $j \in J$ s.t. $A \models_V \gamma$ for all $\gamma \in \Gamma_j$.

Suppose otherwise, that is, for every $j \in J$, there exists $\gamma_j \in \Gamma_j$ s.t. $A \not\models_V \gamma_j$. Let $\Psi$ be the section $\{\gamma_j \mid j \in J\}$; by definition of $\Psi$ we have that $A \not\models_V \psi$ for all $\psi \in \Psi$, which contradicts the assumption that for all $\Psi \in \mathrm{Sec}(\{\Gamma_j \mid j \in J\})$ there exist $t, t' \in T_\Sigma(Var)_{|s}$ s.t. $D(t)$, $D(t')$, $t = t' \in \Psi$, because if $A \not\models_V D(t)$ and $A \not\models_V D(t')$, then $A \models_V t = t'$. $\square$

**Remark**

- Because of rules 2.2b and 5, for all terms, $t, t'$, all sets $\Delta$ of elementary formulas and all elementary formulas $\varepsilon$ we have that $CL(Sp) \vdash \{t = t'\} \cup \Delta \supset \varepsilon$ iff $CL(Sp) \vdash \{t' = t\} \cup \Delta \supset \varepsilon$ and analogously $CL(Sp) \vdash \Delta \supset t = t'$ iff $CL(Sp) \vdash \Delta \supset t' = t$; i.e. $t = t'$ and $t' = t$ are interchangeable and hence in the sequel we shortly use

"...$t = t'$..." instead of "...both $t = t'$ and $t' = t$...", or of "...$t = t'$ or $t' = t$..." and of similar phrases.

- The elimination rule is clearly a generalization of rule $\star$ given above. Now we show an example of the use of elimination rule in an infinitary case.

**spec** $Sp_{10}$ =
**sorts** $s_1, s_2, s_3$
**opns**
$a_j : \rightarrow s_j$, for $j = 1, 2, 3$
$f_i, g_i : s_i \rightarrow s_i$, for $i = 1, 2$
**axioms**
$\alpha_1$ $\{f_1^n(a_1) = g_1^n(a_1) \mid n \in \mathbb{N}\} \supset D(a_3)$
$\alpha_2$ $\{f_2^n(a_2) = g_2^n(a_2) \mid n \in \mathbb{N}\} \supset D(a_3)$
$\alpha_3$ $D_{s_1}(x_1) \wedge D_{s_2}(x_2) \supset D(a_3)$

From $\alpha_3$, by rule 5, we deduce for all $i, j \in \mathbb{N}$:

$$\theta_{i,j}^1 = (D(f_1^i(a_1)) \wedge D(f_2^j(a_2)) \supset D(a_3)),$$

$$\theta_{i,j}^2 = (D(f_1^i(a_1)) \wedge D(g_2^j(a_2)) \supset D(a_3)),$$

$$\theta_{i,j}^3 = (D(g_1^i(a_1)) \wedge D(f_2^j(a_2)) \supset D(a_3)),$$

$$\theta_{i,j}^4 = (D(g_1^i(a_1)) \wedge D(g_2^j(a_2)) \supset D(a_3)).$$

We claim that

$$\Gamma = \{\text{prem}(\alpha_1)\} \cup \{\text{prem}(\alpha_2)\} \cup \{\text{prem}(\theta_{i,j}^k) \mid k = 1, \ldots, 4;\ i, j \in \mathbb{N}\}$$

is ininfluent and hence from $\alpha_1$, $\alpha_2$ and $\{\theta_{i,j}^k \mid k = 1, \ldots, 4;\ i, j \in \mathbb{N}\}$, by the elimination rule, we deduce $D(a_3)$. Indeed let $\Psi$ be a section of $\Gamma$; by definition, in $\Psi$ there are an element of prem$(\alpha_1)$, say $f_1^m(a_1) = g_1^m(a_1)$, an element of prem$(\alpha_2)$, say $f_2^n(a_2) = g_2^n(a_2)$, and an element of prem$(\theta_{n,m}^k)$ for any $k = 1, \ldots, 4$ and hence $\{D(f_1^m(a_1)),\ D(g_1^m(a_1))\} \subseteq \Psi$ or $\{D(f_2^n(a_2)),\ D(g_2^n(a_2))\} \subseteq \Psi$, by definition of prem$(\theta_{n,m}^k)$.

Therefore $\{D(f_1^m(a_1)), D(g_1^m(a_1)), f_1^m(a_1) = g_1^m(a_1)\} \subseteq \Psi$ or $\{D(f_2^n(a_2)), D(g_2^n(a_2)), f_2^n(a_2) = g_2^n(a_2)\} \subseteq \Psi$.

In order to show the seq-completeness of $CL(Sp)$, and also for its independent interest w.r.t. the characterization of the initial models, we first introduce the simpler deduction system $CL_g(Sp)$, where the elimination rule has been restricted to ground formulas. Then we will prove that $CL_g(Sp)$ is seq-complete for $Sp$ and the empty family of variables. Finally, we show that this implies that $CL(Sp)$ is seq-complete.

**Definition 4.4.** Let $Sp$ be $(\Sigma, Ax)$; the inference system $CL_g(Sp)$ (where the index g stands for "ground") consists of the axioms and inference rules of $UL(Sp)$ and of the following inference rule:

*Ground elimination*

$$\frac{\{\Theta_j \cup \Gamma_j \supset \varepsilon \mid j \in J\}}{(\bigcup_{j\in J} \Theta_j) \supset \varepsilon}$$

where
- $\varepsilon \in \text{EForm}(\Sigma, \emptyset)$;
- $\Gamma = \{\Gamma_j \mid j \in J\}$ and $\Theta = \{\Theta_j \mid j \in J\}$ are families of countable subsets of $\text{EForm}(\Sigma, \emptyset)$ indexed by a possibly noncountable $J$;
- $\Gamma$ is ininfluent.

We postpone to the following subsection the (very technical) proofs of the following two fundamental completeness results.

**Theorem 4.5.** *For any conditional specification Sp, the system $CL_g(Sp)$ is seq-complete w.r.t. Sp and the empty family of variables.*

**Proof.** See Section 4.2. □

**Theorem 4.6.** *For any conditional specification Sp, the conditional system $CL(Sp)$ is seq-complete w.r.t. Sp and any family X of variables.*

**Proof.** See Section 4.2. □

From Theorem 3.6 and the completeness of $CL_g(Sp)$, we get the following conclusive results about initiality.

**Theorem 4.7.** *The following conditions are equivalent:*
1. *the set* $\text{NF}(CL_g(Sp))$ *is empty;*
2. *the algebra* $T_\Sigma/{\equiv^{CL_g(Sp)}}$ *is a model of Sp;*
3. *the algebra* $T_\Sigma/{\equiv^{CL_g(Sp)}}$ *is initial in PMod(Sp);*
4. *there exists a model that is initial in PMod(Sp);*
5. *there exists a model that is initial in PGen(Sp).*

**Proof.** By Theorem 4.5, the system $CL_g(Sp)$ is eeq-complete for $Sp$ and the empty family of variables and hence, by Theorem 3.6, we have the thesis. □

Analogously, from Theorem 3.6 and the completeness of $CL(Sp)$, we get the following conclusive results about freeness.

**Theorem 4.8.** *The following conditions are equivalent:*
1. *the set* $\text{NF}(CL(Sp), X)$ *is empty;*
2. *the algebra* $T_\Sigma(X)/{\equiv^{CL(Sp)}}(X)$ *is a model of Sp;*

3. *$(T_\Sigma(X)/\equiv^{CL(Sp)}(X), \mathsf{m}(CL(Sp), X)$ is free for $X$ in $PMod(Sp)$;*
4. *there exists a free object for $X$ in $PMod(Sp)$;*
5. *there exists a free object for $X$ in $PGen(Sp, X)$.*

**Proof.** By Theorem 4.6, the system $CL(Sp)$ is eeq-complete for $Sp$ and $X$ and hence, by Theorem 3.6, we have the thesis. $\square$

### 4.2. Completeness proofs

In the sequel, the seq-completeness of $CL_g(Sp)$ w.r.t the empty family of variables and $Sp$ will be called *gseq-completeness*, and analogously eeq-completeness of $CL_g(Sp)$ w.r.t. the empty family of variables and $Sp$ will be called *geeq-completeness.*

We first prove Theorem 4.5 (gseq-completeness of $CL_g(Sp)$) and then we will show that gseq-completeness of $CL_g(Sp)$ implies seq-completeness of $CL(Sp)$ (Proposition 4.14), thus getting Theorem 4.6.

We first state that $CL_g(Sp)$ satisfies the deduction theorem, which is well known and fundamental in classical logic, w.r.t. elementary ground formulas. The deduction theorem is necessary in the proof of the gseq-completeness theorem and some of the intermediate results.

**Proposition 4.9** (Deduction theorem). *Let $Sp$ be the conditional specification $(\Sigma, Ax)$ and $\Gamma$ be a set of elementary ground formulas over $\Sigma$.*

*Then $CL_g((\Sigma, Ax \cup \Gamma)) \vdash \Delta \supset \varepsilon$ iff $CL_g(Sp) \vdash \Gamma' \cup \Delta \supset \varepsilon$, for some $\Gamma' \subseteq \Gamma$.*

**Proof.** Let $Sp'$ be the conditional specification $(\Sigma, Ax \cup \Gamma)$.

($\Leftarrow$) Let us assume that $CL_g(Sp) \vdash \Gamma' \cup \Delta \supset \varepsilon$. Then, by definition of $Sp$ and $Sp'$, we also have that $CL_g(Sp') \vdash \Gamma' \cup \Delta \supset \varepsilon$. Moreover $CL_g(Sp') \vdash \gamma$ for all $\gamma \in \Gamma'$, since $\Gamma' \subseteq \Gamma$, and hence we also have $CL_g(Sp') \vdash \Delta \supset \varepsilon$, by rule 5, $\Gamma'$ being a set of ground formulas.

($\Rightarrow$) By induction over the definition of $CL_g(Sp')$.

*Proper axioms*

- If $(\Delta \supset \varepsilon) \in Ax$, then $CL_g(Sp) \vdash \Delta \supset \varepsilon$; so the thesis holds for $\Gamma' = \emptyset$;
- If $(\Delta \supset \varepsilon) \in \Gamma$, then $\Delta = \emptyset$; now we show that $CL_g(Sp) \vdash \varepsilon \supset \varepsilon$ for every ground elementary formula $\varepsilon$, and hence we have the thesis for $\Gamma' = \{\varepsilon\}$.
  - Let $\varepsilon$ be the formula $D(t)$, with $t \in T_{\Sigma|_s}$. By rule 4 we have that $CL_g(Sp) \vdash D(t) \wedge t = t \supset D(t)$; by rule 2a we have that $CL_g(Sp) \vdash t = t$; so from rule 5 we also have that $CL_g(Sp) \vdash D(t) \supset D(t)$.
  - Let $\varepsilon$ be the formula $t = t'$, with $t, t' \in T_{\Sigma|_s}$. By rule 2b we have that $CL_g(Sp) \vdash t = t' \supset t' = t$ and $CL_g(Sp) \vdash t' = t \supset t = t'$; so by rule 5 we also have that $CL_g(Sp) \vdash t = t' \supset t = t'$.

*Axioms* 2, …, 4 obvious, for $\Gamma' = \emptyset$.

*Modus ponens.* We assume that $CL_g(Sp') \vdash \Delta \cup \Xi \supset \varepsilon$ and $CL_g(Sp') \vdash \Delta_\xi \supset \xi$ for all $\xi \in \Xi$ so that

$$CL_g(Sp') \vdash D(Var(\Xi) - Var\left(\Delta \cup \bigcup_{\xi \in \Xi} \Delta_\xi \supset \varepsilon\right) \cup \Delta \cup \bigcup_{\xi \in \Xi} \Delta_\xi \supset \varepsilon.$$

By inductive hypothesis, we have that $CL_g(Sp) \vdash \Delta \cup \Xi \cup \Gamma'' \supset \varepsilon$ and $CL_g(Sp) \vdash \Delta_\xi \cup \Gamma'_\xi \supset \xi$ for all $\xi \in \Xi$ and suitable $\Gamma''$, $\Gamma'_\xi \subseteq \Gamma$; thus by rule 5

$$CL_g(Sp) \vdash D(Y) \cup \Delta \cup \Gamma'' \cup \left(\bigcup_{\xi \in \Xi} \Delta_\xi\right) \cup \left(\bigcup_{\xi \in \Xi} \Gamma'_\xi\right) \supset \varepsilon$$

where $Y = Var(\Xi) - Var(\Delta \cup \Gamma'' \cup (\bigcup_{\xi \in \Xi} \Delta_\xi) \cup (\bigcup_{\xi \in \Xi} \Gamma'_\xi) \supset \varepsilon)$ i.e., $\Gamma''$ and $\Gamma'_\xi$ being sets of ground formulas, $Y = Var(\Xi) - Var(\Delta \cup (\bigcup_{\xi \in \Xi} \Delta_\xi) \supset \varepsilon)$; thus we have the thesis for $\Gamma' = \Gamma'' \cup \bigcup_{\xi \in \Xi} \Gamma'_\xi$.

*Instantiation/abstraction.* Assume that $CL_g(Sp') \vdash \Delta \supset \varepsilon$ and denote by $\gamma^*$ the formula $\gamma[t_x/x \mid x \in X_s, s \in S]$; so that

$$CL_g(Sp') \vdash \{D(t_x) \mid x \in X_s, s \in S\} \cup \{\delta^* \mid \delta \in \Delta\} \supset \varepsilon^*.$$

By inductive hypothesis $CL_g(Sp) \vdash \Gamma'' \cup \Delta \supset \varepsilon$, with $\Gamma'' \subseteq \Gamma$. Thus, by rule 6,

$$CL_g(Sp) \vdash \{D(t_x) \mid x \in X_s, s \in S\} \cup \{\delta^* \mid \delta \in \Delta\} \cup \{\gamma^* \mid \gamma \in \Gamma''\} \supset \varepsilon^*$$

i.e. since every $\gamma \in \Gamma''$ is a ground formula,

$$CL_g(Sp) \vdash \{D(t_x) \mid x \in X_s, s \in S\} \cup \Gamma'' \cup \{\delta^* \mid \delta \in \Delta\} \supset \varepsilon^*$$

and hence we have the thesis for $\Gamma' = \Gamma''$.

*Elimination.* Assume that (i) $CL_g(Sp') \vdash \Delta_j \cup \Gamma_j \supset \varepsilon$ for all $j \in J$ and that (ii) there exist $t, t' \in T_\Sigma$ s.t. $D(t), D(t')$ and $t = t'$ belong to $\Psi$ for all $\Psi \in \mathsf{Sec}(\{\Gamma_j \mid j \in J\})$, so that $CL_g(Sp) \vdash (\bigcup_{j \in J} \Delta_j) \supset \varepsilon$. Because of (i) and of the inductive hypothesis $CL_g(Sp) \vdash \Gamma'_j \cup \Gamma_j \cup \Delta_j \supset \varepsilon$, with $\Gamma'_j \subseteq \Gamma$ for all $j \in J$. Thus, by (ii) and elimination rule, $CL_g(Sp) \vdash (\bigcup_{j \in J} \Gamma'_j) \cup (\bigcup_{j \in J} \Delta_j) \supset \varepsilon$, i.e. we have the thesis for $\Gamma' = \bigcup_{j \in J} \Gamma'_j$. □

Note that it is immaterial which variables are used in deduction both by $CL_g(Sp)$ and $CL(Sp)$, since by rules 5, 6 and 1 we are able to replace bound variables. Therefore in the sequel we assume without any loss of generality that all variables used during the proof of any theorem by $CL_g(Sp)$ do not belong to $X$.

**Notation.** We will denote $\mathrm{EEq}(CL_g(Sp), \emptyset)$ by $\mathrm{EEq}(Sp)$, $\mathrm{NF}(CL_g(Sp), \emptyset)$ by $\mathrm{NF}(CL_g(Sp))$ and $\mathsf{Fr}(CL_g(Sp), \emptyset)$ by $I(Sp)$.

We now proceed to show the relative gseq-completeness of $CL_g(Sp)$. In the literature, both in the case of partial positive conditional and of total conditional specifications, the usual way to show that a system $L$ for $Sp$ is geeq-complete is to prove that $I = T_\Sigma/\equiv^L$ is a model, since $I \models \varepsilon$ iff $L \vdash \varepsilon$ for all $\varepsilon \in \mathrm{EEq}(Sp, \emptyset)$. Unfortunately we cannot adopt the same proof technique in the case of (possibly) nonpositive conditional specifications, since we have seen that in general they do not have an initial model (see e.g. theorem 3.6). In other words we cannot give a single model which does not satisfy *all* undeducible ground equalities, but for each undeducible ground equality $\varepsilon$ we have to give a model which does not satisfy $\varepsilon$. To do this, we first restrict ourselves to $\varepsilon \in \mathrm{EEq}(Sp)$ s.t. $CL_g(Sp) \nvdash \varepsilon$ and build a model $A_\varepsilon$ of $Sp$ s.t. $A_\varepsilon \not\models \varepsilon$; in particular we show that there exists an enrichment $Sp_\varepsilon$ of $Sp$ s.t. $CL_g(Sp_\varepsilon) \nvdash \varepsilon$ and $I(Sp_\varepsilon)$ is a model of $Sp_\varepsilon$, so that $A_\varepsilon = I(Sp_\varepsilon) \not\models \varepsilon$ and obviously $A_\varepsilon \in PMod(Sp)$, $Sp_\varepsilon$ being an enrichment of $Sp$.

Then we proof that for any $\eta \notin \mathrm{EEq}(Sp)$ s.t. $CL_g(Sp) \nvdash \eta$ there exists an enrichment $Sp_\eta$ of $Sp$ s.t. $\eta \in \mathrm{EEq}(Sp_\eta)$ and $CL_g(Sp_\eta) \nvdash \eta$, so that we reduce to the above case.

Let us consider now the first problem, the building of such a $Sp_\varepsilon$. Since $I(Sp)$ is not the initial model of $Sp$ iff $\mathrm{NF}(CL_g(Sp))$ is nonempty, we try to clear out $\mathrm{NF}(CL_g(Sp))$ by adding as axioms of the enrichment suitable elementary formulas that make false $nf_2$ or $nf_3$ in the enriched specification for all formulas in $\mathrm{NF}(CL_g(Sp))$ (we call the set of these formulas a "resolving choice"). Obviously it is possible that in the enriched specification a conditional formula is naughty, which in $Sp$ was not, because it was not an instantiation of an axiom by *defined* terms, or because one of its premises was not deducible; so that also the enriched specification does not have an initial model. Therefore we have to consider a wider class of formulas, the "possibly naughty" formulas.

**Definition 4.10.**

- For a given conditional specification $Sp$, the set $\mathrm{PNF}(Sp)$ (for Possibly Naughty Formulas) consists of all ground conditional formulas $\phi$ s.t.
  - $CL_g(Sp) \vdash \phi$;
  - $CL_g(Sp) \nvdash \mathrm{cons}(\phi)$.
- An *r-choice* (for resolving choice) $C$ is a set of ground elementary formulas s.t. for all $\phi \in \mathrm{PNF}(Sp)$
  If $(\mathrm{prem}(j) \cap \mathrm{EEq}(Sp)) \subseteq C$, then $\mathrm{cons}(j) \in C$, or there exists $(t = t') \in \mathrm{prem}(j) - \mathrm{EEq}(Sp)$ s.t. $(t = t')$, $(t' = t) \notin C$ and $D(t)$ or $D(t')$ belongs to $C$.
- The set of all $r$-choices is denoted by *R-Choice*.

Note that there is always at least one $r$-choice: the set of all elementary formulas.

The first intermediate result that we need is that the resolving choices are really resolving, i.e. that the enriched specification have initial model.

**Lemma 4.11.** *For all conditional specifications* $Sp = (\Sigma, Ax)$ *and all r-choices* $C$, $\mathrm{NF}(\Sigma, Ax \cup C) = \emptyset$.

**Proof.** Let $C$ be an r-choice for $Sp$ and $Sp'$ be the conditional specification $(\Sigma, Ax \cup C)$; in order to show that $\mathrm{NF}(Sp')$ is empty, we assume that the conditions $nf_1$ and $nf_2$ (cf. Definition 2.5) hold for a conditional formula $\phi$ and the system $CL_g(Sp')$ and show that the condition $nf_3$ does not hold. In particular we show that either $CL_g(Sp) \vdash \mathrm{cons}(\phi)$ or $\mathrm{cons}(\phi)$ belongs to $C$.

From $nf_1$, because of rules 5 and 6, $CL_g(Sp') \vdash \phi$ and hence, because of deduction theorem, there exists a subset $\Gamma$ of $C$ s.t. $CL_g(Sp) \vdash \Gamma \cup \mathrm{prem}(\phi) \supset \mathrm{cons}(\phi)$. Moreover, because of $nf_2$, $I(Sp') \models \delta$ for all $\delta \in \mathrm{prem}(\phi)$ and hence, by definition of $I(Sp')$, $CL_g(Sp') \vdash \delta$ for all $\delta \in \mathrm{prem}(\phi) \cap \mathrm{EEq}(Sp')$. Thus, because of deduction theorem, for all $\delta \in \mathrm{prem}(\phi) \cap \mathrm{EEq}(Sp')$ there exists a subset $\Gamma_\delta$ of $C$ s.t. $CL_g(Sp) \vdash \Gamma_\delta \supset \delta$.

Thus, because of rule 5, $CL_g(Sp) \vdash \phi'$, where $\phi'$ is the conditional formula

$$\left(\Gamma \cup \bigcup_{\delta \in \mathrm{prem}(\phi) \cap \mathrm{EEq}(Sp')} \Gamma_\delta\right) \cup (\mathrm{prem}(\phi) - \mathrm{EEq}(Sp')) \supset \mathrm{cons}(\phi).$$

Therefore either $CL_g(Sp) \vdash \mathrm{cons}(\phi)$ or $\phi' \in \mathrm{PNF}(Sp)$. Assume that $\phi' \in \mathrm{PNF}(Sp)$ and show that $\mathrm{cons}(\phi) \in C$. By definition of $Sp$ and $Sp'$ we have $\mathrm{EEq}(Sp) \subseteq \mathrm{EEq}(Sp')$ and hence $\mathrm{prem}(\phi') \cap \mathrm{EEq}(Sp) \subseteq \mathrm{prem}(\phi') \cap \mathrm{EEq}(Sp')$; moreover, by definition of $\phi'$, $\mathrm{prem}(\phi') \cap \mathrm{EEq}(Sp') \subseteq (\Gamma \cup \bigcup_{\delta \in \mathrm{prem}(\phi) \cap \mathrm{EEq}(Sp')} \Gamma_\delta) \subseteq C$, so that $\mathrm{prem}(\phi') \cap \mathrm{EEq}(Sp) \subseteq C$. Thus, by the definition of $r$-choice, $\mathrm{cons}(\phi') \in C$, or there exists $(t = t') \in \mathrm{prem}(\phi') - \mathrm{EEq}(Sp)$ s.t. $(t = t'), (t' = t) \notin C$ and $D(t)$ or $D(t')$ belongs to $C$.

Finally, by definition of $\phi'$, for any $(t = t') \in (\mathrm{prem}(\phi') - \mathrm{EEq}(Sp))$, $(t = t') \in (\Gamma \cup \bigcup_{\delta \in \mathrm{prem}(\phi) \cap \mathrm{EEq}(Sp')} \Gamma_\delta)$ and hence $(t = t') \in C$, or $(t = t') \in (\mathrm{prem}(\phi) - \mathrm{EEq}(Sp'))$. Thus, by definition of $\mathrm{EEq}(Sp')$, neither $D(t) \in C$ nor $D(t') \in C$. Therefore, by the definition of r-choice, $\mathrm{cons}(\phi') \in C$. □

Thus we have that for all r-choices $C$, the enriched specification has an initial model; now we claim that for any $\varepsilon \in \mathrm{EEq}(Sp)$ s.t. $CL_g(Sp) \nvdash \varepsilon$ there exists at least one r-choice s.t. also in the enriched specification $\varepsilon$ does not hold.

**Lemma 4.12.** *If $Sp = (\Sigma, Ax)$ is a conditional specification and $\varepsilon$ is an elementary ground formula s.t. $CL_g(Sp) \nvdash \varepsilon$, then there exists an r-choice $C$ s.t. $CL_g((\Sigma, Ax \cup C)) \nvdash \varepsilon$.*

**Proof.** Let us assume by contradiction that $CL_g((\Sigma, Ax \cup C)) \vdash \varepsilon$ for all r-choices $C$ and show that there exists a set $\Theta$ of conditional formulas s.t.

1. $\mathrm{cons}(\theta) = \varepsilon$ for all $\theta \in \Theta$;
2. $CL_g(Sp) \vdash \theta$ for all $\theta \in \Theta$;
3. for every $\Psi \in \mathrm{Sec}(\{\mathrm{prem}(\theta) \mid \theta \in \Theta\})$ there exist $t, t' \in T_{\Sigma|_s}$ s.t. $D(t)$, $D(t')$ and $t = t'$ belong to $\Psi$.

Thus, by the elimination rule, we have $CL_g(Sp) \vdash \varepsilon$, contrary to the hypothesis. By the assumption that $CL_g((\Sigma, Ax \cup C)) \vdash \varepsilon$ for all r-choices $C$ and the deduction theorem, we have that for each r-choice $C$ a $\Gamma_C \subseteq C$ exists s.t. $CL_g(Sp) \vdash \Gamma_C \supset \varepsilon$; let $\Theta$ be the set $\bigcup_{C \in R-Choice} R(\Gamma_C \supset \varepsilon)$, where $R(\Gamma_C \supset \varepsilon)$ is the set

$$\left\{ \Gamma_1 \cup \bigcup_{\gamma \in \Gamma_2} \Delta_\gamma \supset \varepsilon \mid \Gamma_1 \cup \Gamma_2 = \Gamma_C, \Gamma_1 \cap \Gamma_2 = \emptyset, \forall \gamma \in \Gamma_2 (\Delta_\gamma \supset \gamma) \in \mathrm{PNF}(Sp) \right\}$$

Since $CL_g(Sp) \vdash \psi$ for all $\psi \in \mathrm{PNF}(Sp)$ and $CL_g(Sp) \vdash \Gamma_C \supset \varepsilon$ for all r-choices $C$, then, by rule 5, $CL_g(Sp) \vdash \theta$ for $\theta \in \Theta$; thus $\Theta$ satisfies conditions 1 and 2 and hence we only have to show that $\Theta$ satisfies also condition 3.

Again the proof is by contradiction: assume that condition 3 does not hold, i.e. that there exists $\Psi \in \mathrm{Sec}(\{\mathrm{prem}(\theta) \mid \theta \in \Theta\})$ s.t. for all $t, t' \in T_{\Sigma|_s}$ at least one among $D(t)$, $D(t')$ and $t = t'$ does not belong to $\Psi$, and show that there exists $\theta \in \Theta$ s.t. $\mathrm{prem}(\theta) \cap \Psi = \emptyset$, contrary to the assumption that $\Psi \in \mathrm{Sec}(\{\mathrm{prem}(\theta) \mid \theta \in \Theta\})$.

In order to prove that there exists such a $\theta$ we built an r-choice $C$ s.t. for all $\gamma \in C$ either $\gamma \notin \Psi$, or there exists $(\Delta_\gamma \supset \gamma) \in \mathrm{PNF}(Sp)$ s.t. $\Delta_\gamma \cap \Psi = \emptyset$; thus $\theta' = [(\Gamma_C - \Psi) \cup \bigcup_{\gamma \in \Gamma_C \cap \Psi} \Delta_\gamma] \supset \varepsilon$ belongs to $R(\Gamma_C \supset \varepsilon)$ and hence to $\Theta$, while $\mathrm{prem}(\theta') \cap \Psi = \emptyset$.

Let $C$ be $C_C \cup C_P$, for $C_C = \{\gamma \mid (\Delta \supset \gamma) \in \mathrm{PNF}(Sp), \Delta \cap \Psi = \emptyset\}$ and $C_P = \{D(t) \mid t \in T_\Sigma, D(t) \notin \Psi\}$.

We only have to show that $C$ is a choice. Let $\chi$ belong to $\mathrm{PNF}(Sp)$ s.t. $\mathrm{prem}(\chi) \cap \mathrm{EEq}(Sp) \subseteq C$; then, by definition of choice, we have to show that $\mathrm{cons}(\chi) \in C$, or $*$ [there exists $(t = t') \in \mathrm{prem}(\chi) - \mathrm{EEq}(Sp)$ s.t. $(t = t') \notin C$ and $D(t)$ or $D(t')$ belongs to $C$].

Thus assume that $\star$ does not hold, i.e. that for all $(t = t') \in \mathrm{prem}(\chi) - \mathrm{EEq}(Sp)$ $(t = t') \in C$, or both $D(t)$ and $D(t')$ do not belong to $C$, and show that there exists $\chi' \in \mathrm{PNF}(Sp)$ s.t. $\mathrm{prem}(\chi') \cap \Psi = \emptyset$ and $\mathrm{cons}(\chi') = \mathrm{cons}(\chi)$, so that $\mathrm{cons}(\chi') \in C_C$ and hence $\mathrm{cons}(\chi) \in C$. Since $CL_g(Sp) \vdash \psi$ for all $\psi \in \mathrm{PNF}(Sp)$, $R(\psi) \subseteq \mathrm{PNF}(Sp)$ for all $\psi \in \mathrm{PF}(Sp)$; so that in order to build such a $\chi'$ it is sufficient to show that for all $\eta \in \mathrm{prem}(\chi) \cap \Psi$ there exists $(\Delta_\eta \supset \eta) \in \mathrm{PNF}(Sp)$ s.t. $\Delta_\eta \cap \Psi = \emptyset$ and we have the thesis for $\chi' = [(\mathrm{prem}(\chi) - \Psi) \cup (\bigcup_{\eta \in (\mathrm{prem}(\chi) \cap \Psi)} \Delta_\eta)] \supset \mathrm{cons}(\chi)$.

Let $\eta \in \mathrm{prem}(\chi) \cap \Psi$; we show that $\eta \in C$ and hence $\eta \in C \cap \Psi = C_C$ so that, by definition of $C$, there exists $(\Delta_\eta \supset \eta) \in \mathrm{PNF}(Sp)$ s.t. $\Delta_\eta \cap \Psi = \emptyset$. If $\eta \in \mathrm{EEq}(Sp)$, then, since we have assumed that $\mathrm{prem}(\chi) \cap \mathrm{EEq}(Sp) \subseteq C$, $\eta \in C$. Otherwise $\eta \in \mathrm{prem}(\chi) - \mathrm{EEq}(Sp)$ has the form $t = t'$ and, because of the absurd hypothesis on $\Psi$ and $t = t' \in \Psi$, $D(t)$ or $D(t')$ does not belong to $\Psi$, so that, because of definition of $C_P$, $D(t)$ or $D(t')$ belongs to $C$; thus, since we have assumed that $(t = t') \in C$, or both $D(t)$ and $D(t')$ do not belong to $C$, $\eta = (t = t') \in C$. □

We are now able to show the gseq-completeness of $CL_g(Sp)$ (Theorem 4.5).

**Proof of Theorem 4.5.** Let $\varepsilon$ be an elementary ground formula and assume that $CL_g(Sp) \nvdash \varepsilon$; we show that there exists a model $A$ of $Sp = (\Sigma, Ax)$ s.t. $A \not\models \varepsilon$. We divide the proof in two cases.

- Let $\varepsilon$ belong to $\mathrm{EEq}(Sp)$)
  - If $\mathrm{NF}(CL_g(Sp))$ is empty, then $A = T_\Sigma/\equiv^{CL_g(Sp)}$ is a model, because of the theorem 3.6, and, by construction of $\equiv^{CL_g(Sp)}$, $A \not\models \varepsilon$.
  - Otherwise, by Lemma 4.12, there exists an r-choice $C$ s.t. $CL_g(\Sigma, Ax \cup C)) \not\vdash \varepsilon$. Moreover, by Lemma 4.11, $\mathrm{NF}(CL_g(Sp'))$ is empty, where $Sp' = (\Sigma, Ax \cup C)$. Thus, by Theorem 3.6, $A = T_\Sigma/\equiv^{CL_g(Sp')}$ is a model of $Sp'$. Finally, $A$ belongs to $PMod(Sp)$, since $PMod(Sp') \subseteq PMod(Sp)$ by definition of $Sp'$, and moreover $A \not\models \varepsilon$, by definition of $A$.
- Let $\varepsilon$ have the form $t = t'$, $CL_g(Sp) \not\vdash D(t)$, and $CL_g(Sp) \not\vdash D(t')$; if there exists a conditional specification $Sp'$ s.t.
  - $PMod(Sp') \subseteq PMod(Sp)$
  - $\varepsilon \in \mathrm{EEq}(CL_g(Sp'))$
  - $CL_g(Sp') \not\vdash \varepsilon$,

  then there exists a model $A$ of $Sp'$ s.t. $A \not\models \varepsilon$ and hence $A$ is also a model of $Sp$ which does not satisfy $\varepsilon$. Therefore we only have to show that there exists such a $Sp'$.
  Let $Sp_1$ be the specification $(\Sigma, Ax \cup \{D(t)\})$ and $Sp_2$ be the specification $(\Sigma A\, Ax \cup \{D(t')\})$; we show that $CL_g(Sp_1) \not\vdash t = t'$ or $CL_g(Sp_2) \not\vdash t = t'$. By contradiction, assume that $CL_g(Sp_1) \vdash t = t'$ and $CL_g(Sp_2) \vdash t = t'$. Then we prove that $CL_g(Sp) \vdash t = t'$. Because of the absurd hypothesis and of the deduction theorem, we have both $CL_g(Sp) \vdash D(t) \supset t = t'$ and $CL_g(Sp) \vdash D(t') \supset t = t'$; moreover, by rule 2b, $CL_g(Sp) \vdash t = t' \supset t' = t$ and $CL_g(Sp) \vdash t' = t \supset t = t'$ and hence, by rule 5, $CL_g(Sp) \vdash t = t' \supset t = t'$. Thus applying the elimination rule to the set $\{D(t) \supset t = t', D(t') \supset t = t', t = t' \supset t = t'\}$, we have $CL_g(Sp) \vdash t = t'$. Therefore $CL_g(Sp_1) \not\vdash t = t'$, and in this case let $Sp'$ be $Sp_1$, or $CL_g(Sp_2) \not\vdash t = t'$, and in this case let $Sp'$ be $Sp_2$. □

We now show that the gseq-completeness of $CL_g(Sp)$ implies the seq-completeness of $CL(Sp)$ and thus we complete the proof of Theorem 4.6.

**Lemma 4.13.** *Let $\Sigma = (S, F)$ be a signature, $Sp = (\Sigma, Ax)$ be a conditional specification and $X$ be an S-sorted family of variables.*

*Let $Sp_X$ denote the conditional specification $(\Sigma_X, Ax_X)$, where $\Sigma_X = (S, F \cup \{op_x : \to s \mid x \in X_s\}_{s \in S})$ and $Ax_X = Ax \cup \{D(op_x) \mid x \in X\}$, and $\phi^*$ denote $\phi[op_x/x \mid x \in X]$ for all conditional formulas $\phi$.*

*For all conditional formulas $\Delta \supset \varepsilon$ we have that $CL_g(Sp_X) \vdash \Delta^* \supset \varepsilon^*$ implies $CL(Sp) \vdash D(X) \cup \Delta \supset \varepsilon$.*

**Proof.** Since rule 6 allows one to arbitrary increase the definedness assertions in the premises of any deduced formula, it is sufficient to prove that $CL_g(Sp_X) \vdash \Delta^* \supset \varepsilon^*$ implies $CL(Sp) \vdash D(X') \cup \Delta \supset \varepsilon$ for some $X' \subseteq X$. The proof is done by induction on the definition of $CL_g(Sp_X)$.

*Proper axioms*

- If $(\Delta \supset \varepsilon) \in Ax$, then $CL(Sp) \vdash \Delta \supset \varepsilon$, by definition.
- Otherwise $(\Delta \supset \varepsilon) = D(op_x)$ and we have the thesis because of rule 1.

*Axioms* 2, ..., 4 obvious.

*Modus ponens.* Let us assume that $CL_g(Sp_X) \vdash \Theta^* \cup \Gamma^* \supset \varepsilon^*$ and $CL_g(Sp_X) \vdash \Theta^*_\gamma \supset \gamma^*$ for all $\gamma \in \Gamma$; then

$$CL_g(Sp_X) \vdash D(Z) \cup \Theta^* \cup \bigcup_{\gamma \in \Gamma_2} \Theta^*_\gamma \supset \varepsilon^*$$

where $Z = Var(\Gamma^*) - Var(\Theta^* \cup (\bigcup_{\gamma \in \Gamma} \Theta^*_\gamma) \supset \varepsilon^*)$. Since $Z \cap X = \emptyset$ by definition of $Z$, we have to show that

$$CL(Sp) \vdash D(X) \cup D(Z) \cup \Theta \cup \left( \bigcup_{\gamma \in \Gamma} \Theta_\gamma \right) \supset \varepsilon.$$

By inductive hypothesis, $CL(Sp) \vdash D(X) \cup \Theta \cup \Gamma \supset \varepsilon$ and $CL(Sp) \vdash D(X) \cup \Theta_\gamma \supset \gamma$ for all $\gamma \in \Gamma$. Thus $CL(Sp) \vdash D(X) \cup D(Y) \cup \Theta \cup (\bigcup_{\gamma \in \Gamma} \Theta_\gamma) \supset \varepsilon$, because of rule 5 of $CL(Sp)$, where $Y$ is the set $Var(\Gamma) - Var(\Theta \cup (\bigcup_{\gamma \in \Gamma} \Theta_\gamma) \supset \varepsilon)$. Moreover $Y \cup X = Z \cup X$ by definition of $Y$ and $Z$ and hence we have the thesis.

*Instantiation/abstraction.* Let us assume that $CL_g(Sp_X) \vdash \Delta^* \supset \varepsilon^*$; then for all families $Z$ of variables (for $CL_g(Sp)$) and all $\Gamma_{\mathrm{Def}} = \{D(t_z) \mid z \in Z\}$, $CL_g(Sp_X) \vdash \phi$, where $\phi$ is

$$\Gamma_{\mathrm{Def}} \cup \{\delta[t_z/z \mid z \in Z] \mid \delta \in \Delta^*\} \supset \varepsilon^*[t_z/z \mid z \in Z].$$

For all $t_z$ one term $t'_z$ exists s.t. $t'^*_z = t_z$; thus $(\eta^*[t_z/z \mid z \in Z])^* = (\eta[t'_z/z \mid z \in Z])^*$ for all elementary formulas $\eta$ and hence $\phi$ is also

$$(\Gamma'_{\mathrm{Def}})^* \cup (\{\delta[t'_z/z \mid z \in Z] \mid \delta \in \Delta\})^* \supset (\varepsilon[t_z/z \mid z \in Z])^*.$$

where $\Gamma'_{\mathrm{Def}}$ is $\{D(t'_z) \mid z \in Z\}$. By inductive hypothesis, we have that $CL(Sp) \vdash D(X) \cup \Delta \supset \varepsilon$; thus, because of rule 6 of $CL(Sp)$, $CL(Sp) \vdash \phi'$, where $\phi'$ is

$$\Gamma'_{\mathrm{Def}} \cup \{\delta[t'_z/z \mid z \in Z] \mid \delta \in \Delta \cup D(X)\} \supset \varepsilon[t'_z/z \mid z \in Z].$$

Finally, since $X \cap Z = \emptyset$, because we have assumed that all variables in $X$ do not appear in any proof of $CL_g(Sp)$, we have that $D(x)[t_z/z \mid z \in Z] = D(x)$ for all $x \in X$, and hence

$$\{\delta[t_z/z \mid z \in Z] \mid \delta \in \Delta \cup D(X)\} = D(X) \cup \{\delta[t_z/z \mid z \in Z] \mid \delta \in \Delta\}$$

so that $\phi'$ is

$$D(X) \cup \Gamma'_{\mathrm{Def}} \cup \{\delta[t'_z/z \mid z \in Z] \mid \delta \in \Delta\} \supset \varepsilon[t'_z/z \mid z \in Z]$$

and hence we have the thesis.

*Elimination.* Assume that $CL_g(Sp_X) \vdash \Delta^*_j \cup \Gamma^*_j \supset \varepsilon^*$ for all $j \in J$ and that for all $\Psi \in \mathrm{Sec}(\{\Gamma^*_j \mid j \in J\})$ there exist $t, t' \in T_\Sigma(X)$ s.t. $D(t)$, $D(t')$ and $t = t'$ belong to $\Psi$.

Then, by inductive hypothesis, $CL(Sp) \vdash D(X) \cup \Delta_j \cup \Gamma_j \supset \varepsilon$ for all $j \in J$ and, by definition of $\Gamma_j^*$, for all $\Psi \in \mathrm{Sec}(\{\Gamma_j \mid j \in J\})$ there exist $t, t' \in T_\Sigma(X)$ s.t. $D(t)$, $D(t')$ and $t = t'$ belong to $\Psi$. Thus, by elimination rule of $CL(Sp)$, we have $CL(Sp) \vdash D(X) \cup (\bigcup_{j \in J} \Delta_j) \supset \varepsilon$. □

**Proposition 4.14.** *If $CL_g(Sp)$ is* gseq-*complete for all conditional specifications $Sp$, then $CL(Sp')$ is* seq-*complete for all families of variables and all conditional specifications $Sp'$.*

**Proof.** Let $\Sigma = (S, F)$ be a signature, $Sp = (\Sigma, Ax)$ be a conditional specification and $X$ be an $S$-sorted family of variables. Using the notation of Lemma 4.13, we show that the gseq-completeness of $CL_g(Sp_X)$ implies the seq-completeness of $CL(Sp)$ for $X$ and $Sp$.

Let us assume that $CL_g(Sp_X)$ is gseq-complete, $\varepsilon$ is an elementary formula over $\Sigma = (S, F)$ and $X$ s.t. $CL(Sp) \nvdash D(X') \supset \varepsilon$ for any $X' \subseteq X$. We show that there exists a model $A$ s.t. $A \not\models D(X) \supset \varepsilon$, i.e. that there exists a valuation $V: X \to A$ s.t. $A \not\models_V \varepsilon$. To this end is is sufficient to prove that there exists a model $B$ of $Sp_X$ s.t. $B \not\models \varepsilon^*$; indeed we have the thesis for $A$ defined by $s^A = s^B$ for all $s \in S$, $op^A = op^B$ for all $op \in F$ (i.e. $A$ is the $\Sigma$-reduct of $B$) and $V$ defined by $V(x) = {op_x}^B$, which is well defined because of the axioms $D(op_x)$. In order to prove that there exists such a model $B$, $CL_g(Sp_X)$ being gseq-complete, it is sufficient to show that $CL_g(Sp_X) \nvdash \varepsilon^*$. By Lemma 4.13, if $CL_g(Sp_X) \vdash \varepsilon^*$, then $CL(Sp) \vdash D(X) \supset \varepsilon$ and hence, since we have assumed that $CL(Sp) \nvdash D(X) \supset \varepsilon$, we conclude that $CL_g(Sp_X) \nvdash \varepsilon^*$. □

Finally we get the seq-completeness result for $CL(Sp)$.

**Proof of Theorem 4.6.** From Proposition 4.14, because, by Theorem 4.5, $CL_g(Sp')$ is gseq-complete for all conditional specifications $Sp'$. □

### *4.3. A complete system for the finitary case*

If all the axioms of a conditional specification $Sp$ have a finite set of premises, then we can specialize the system $CL(Sp)$ and obtain a seq-complete system whose rules have a finitary number of premises and only deal with finitary formulas. In this case the elimination rule takes a very simple and intuitive form (which was already illustrated in Section 4.1).

**Definition 4.15.** Let $Sp = (\Sigma, Ax)$ be a conditional specification s.t. $\Delta$ is finite for all $\Delta \supset \varepsilon$ in $Ax$. Let $CL_f(Sp)$ be the inference system consisting of the axioms in $Ax$, of the axioms $1, \ldots, 4$ of $CL(Sp)$ (definedness of variables, congruence, strictness, definedness and equality) and of the following inference rules, where we assume that any formula is finitary:

$5_f$. *Modulus ponens*

$$\frac{\Delta \cup \{\gamma\} \supset \varepsilon,\ \Delta_\gamma \supset \gamma}{D(Var(\gamma) - Var(\Delta \cup \Delta_\gamma \supset \varepsilon)) \cup (\Delta \cup \Delta_\gamma) \supset \varepsilon}$$

$6_f$. *Instantiation/abstraction*

$$\frac{\Delta \supset \varepsilon}{\{D(t)\} \cup \{\delta[t/x] \mid \delta \in \Delta\} \supset \varepsilon[t/x]}$$

where $t \in T_\Sigma(Var)_{|_s}$, $x \in X_s$.

$7_f$. *Elimination*

$$\frac{(\Delta_1 \cup \{D(t)\}) \supset \varepsilon,\ (\Delta_2 \cup \{D(t')\}) \supset \varepsilon,\ (\Delta_3 \cup \{t = t'\}) \supset \varepsilon}{D(Var(t = t')) \cup (\Delta_1 \cup \Delta_2 \cup \Delta_3) \supset \varepsilon}$$

The seq-completeness of $CL_f(Sp)$ follows from the seq-completeness of $CL(Sp)$, because $CL(Sp) \vdash \Delta \supset \varepsilon$ implies that there exists a finite $\Gamma \subseteq \Delta$ s.t. $CL_f(Sp) \vdash \Gamma \supset \varepsilon$. To prove this claim we need two intermediate results. The first lemma is stated in a merely combinatorial form and guarantees that any application of the elimination rule to a possibly infinite set of finitary premises may be replaced by an application of the elimination rule to a *finite* set of finitary premises. The second lemma states that an application of the elimination rule to a *finite* set of finitary premises can be replaced by a *finite* sequence of application of the rule of finitary elimination $7_f$.

**Notation.** Let Val be a denumerable set of values, $\mathcal{R} \subseteq \wp_{Fin}(\mathsf{Val})$ be a relation over Val and $\Gamma$ be a (possibly more than denumerable) collection of subsets of Val. We denote by $\mathsf{Sec}(\Gamma)$ the collection of all possible *sections* of $\Gamma$, i.e. of all subsets $\sigma$ of Val s.t. $\sigma = \{v_\gamma \mid \gamma \in \Gamma\}$ for some $v_\gamma \in \gamma$ for each $\gamma \in \Gamma$, and say that $\Gamma$ is *$\mathcal{R}$-ininfluent* iff for any section $\sigma$ of $\Gamma$ there exists $R \in \mathcal{R}$ s.t. $R \subseteq \sigma$.

**Lemma 4.16.** *Let* Val *be a denumerable set of values, $\mathcal{R} \subseteq \wp_{Fin}(\mathsf{Val})$ be a relation over* Val *and $\Gamma$ be a (possibly more than denumerable) collection of subsets of* Val. *If $\Gamma$ is an $\mathcal{R}$-ininfluent collection of subsets of* Val *s.t. $\gamma$ is finite for all $\gamma \in \Gamma$, then there exists an $\mathcal{R}$-ininfluent finite subset of $\Gamma$.*

**Proof.** If $\Gamma$ is finite, then we trivially have the thesis; thus assume that $\Gamma$ is infinite. Since Val is denumerable and $\Gamma$ is a collection of finite subsets of Val, $\Gamma$ is denumerable, too. Let us fix a numeration for $\Gamma$ and denote $\Gamma = \{\Gamma_i \mid i \in \mathbb{N}\}$. Now we build a tree whose finite paths are all and only the sections of $\{\Gamma_i \mid i \leqslant n\}$ for all $n \in \mathbb{N}$ which do not contain any element of $\mathcal{R}$ and show that this tree must be finite. Let us inductively define the tree.

$T_0$ is just the root, labeled by the empty set.

$T_{n+1}$ is the tree obtained from $T_n$ by the following rule:

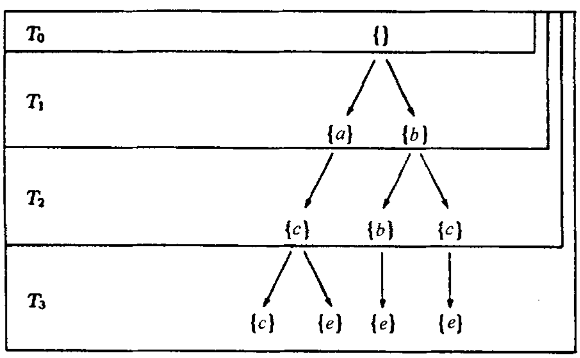
for any leaf $l$ of $T_n$ let us denote by $path(l)$ the set of the labels of the nodes from the root to $l$; then to each leaf $l$ of $T_n$ at depth $n$ we add a son labeled $\{\gamma\}$ for all $\gamma \in \Gamma_n$ s.t. $path(l) \cup \{\gamma\}$ does not contain elements of $\mathscr{R}$.

(Note that in this way if at step $k$ we have not added a son to some leaf, then at any further step, say $n$, this leaf is at depth $k < n$ and hence we do not add sons to it any more).

Let us assume by contradiction that there does not exist $n \in \mathbb{N}$ s.t. $T_n = T_{n+1}$. Then by construction we get an infinite tree, which is finitely branching, because any $\Gamma_n$ is finite; hence, by the König's lemma, there exists an infinite path. But an infinite path, by construction, should be a section for $\Gamma$ which does not contain any element of $\mathscr{R}$, contrary to the assumption that $\Gamma$ is $\mathscr{R}$-ininfluent. Therefore there exists $\bar{n} \in \mathbb{N}$ s.t. $T_{\bar{n}} = T_{\bar{n}+1}$. Then $\Gamma' = \{\Gamma_0, \ldots, \Gamma_{\bar{n}+1}\}$ is $\mathscr{R}$-ininfluent. Indeed let $\{\gamma_0, \ldots, \gamma_{\bar{n}+1}\}$ be a section for $\Gamma'$; then either there exists $R \in \mathscr{R}$ s.t. $R \subseteq \{\gamma_0, \ldots, \gamma_{\bar{n}}\}$ or $\{\gamma_0, \ldots, \gamma_{\bar{n}+1}\}$ is $path(l)$ for some leaf $l$ of $T_{\bar{n}}$ and hence for each $\gamma \in \Gamma_{\bar{n}+1}$ there exists $R \in \mathscr{R}$ s.t. $R \subseteq \{\gamma_0, \ldots, \gamma_{\bar{n}}, \gamma\}$, because $T_{\bar{n}} = T_{\bar{n}+1}$, so that in particular $\{\gamma_0, \ldots, \gamma_{\bar{n}+1}\}$ contains an element of $\mathscr{R}$. Thus in both cases there exists $R \in \mathscr{R}$ s.t. $R \subseteq \{\gamma_0, \ldots, \gamma_{\bar{n}+1}\}$ and hence $\Gamma'$ is $\mathscr{R}$-ininfluent. $\square$

Let us see a simple example of such trees $T_n$.

- $\mathsf{Val} = \{a, b, c, d, e, \ldots\}$;
- $\mathscr{R} = \{\{a, b\}, \{a, c, d\}, \{b, d\}\}$;
- $\Gamma_0 = \{a, b\}$; $\Gamma_1 = \{b, c\}$; $\Gamma_2 = \{a, d, e\}$; ...




In order to show that an application of the elimination rule to a *finite* set of finitary premises can be replaced by a *finite* sequence of application of the rule of finitary elimination $7_f$, we need a preliminary result.

**Lemma 4.17.** *Let $\Theta_j \cup \Gamma_j \supset \varepsilon$ be conditional formulas for $j = 1, \ldots, k$; if the following conditions* 1 *and* 2 *are satisfied, then $CL_f(Sp) \vdash D(Y) \cup \Theta \supset \varepsilon$ holds for some finite $\Theta \subseteq \bigcup_{j=1,\ldots,k} \Theta_j$ and $Y \subseteq Var(\bigcup_{j=1,\ldots,k} \Gamma_j)$.*

1. *$CL_f(Sp) \vdash \Theta_j \cup \Gamma_j \supset \varepsilon$ for $j = 1, \ldots, k$;*

2. *there exist terms* $t_i$, $t'_i$ *for* $i = 1, \ldots, n$ *s.t. for each section* $\sigma$ *of* $\mathscr{R} = \{\mathscr{R}(t_i, t'_i) \mid i = 1, \ldots, n\}$, *where* $R(t, t') = \{D(t), D(t'), t = t'\}$, *there exists* $j \in J$ *s.t.* $\Gamma_j \subseteq \sigma$.

**Proof.** The proof is done by induction over $n$.

- Let $n$ be 1; then from condition 2 either there exists $j \in \{1, \ldots, k\}$ s.t. $\Gamma_j = \emptyset$, and in this case $CL_f(Sp) \vdash \Theta_j \supset \varepsilon$, because of condition 1, so we conclude for $\Theta = \Theta_j$ and $Y = \emptyset$, or there are $j_1, j_2, j_3$ s.t. $\Gamma_{j_1} = \{D(t_1)\}$, $\Gamma_{j_2} = \{D(t'_1)\}$, $\Gamma_{j_3} = \{t_1 = t'_1\}$ and in this case, because of rule $7_f$, $CL_f(Sp) \vdash D(Var(t_1 = t'_1)) \cup (\Theta_{j_1} \cup \Theta_{j_2} \cup {}_{j_3}) \supset \varepsilon$, so we conclude for $X = Var(t_1 = t'_1)$ and $\Theta = \Theta_{j_1} \cup \Theta_{j_2} \cup \Theta_{j_3}$.
- Let us assume that for all conditional formulas $\Theta'_j \cup \Gamma'_j \supset \varepsilon'$ for $j = 1, \ldots, k'$ satisfying conditions 1 and 2 (for $i = 1, \ldots, n$) there exist $\Theta' \subseteq \bigcup_{j=1,\ldots,k'} \Theta'_j$ and $Y' \subseteq Var(\bigcup_{j=1,\ldots,k'} \Gamma'_j)$ s.t. $CL_f(Sp) \vdash D(Y') \cup \Theta' \supset \varepsilon'$ and that $\{\Theta_j \cup \Gamma_i \supset \varepsilon \mid j = 1, \ldots, k\}$ satisfies conditions 1 and 2 for $n + 1$.

  Then for any section $\sigma$ of $\mathscr{R} = \{R(t_i, t'_i) \mid i = 1, \ldots, n\}$ let us define the sets $\Delta_\sigma$ and $\Psi_\sigma$, as follows. Let $\sigma_1$ be $\sigma \cup \{D(t_{n+1})\}$, $\sigma_2$ be $\sigma \cup \{D(t'_{n+1})\}$ and $\sigma_3$ be $\sigma \cup \{t_{n+1} = t'_{n+1}\}$; because of condition 2, there exist $j_1, j_2, j_3 \in \{1, \ldots, k\}$ s.t. $\Gamma_{j_i} \subseteq \sigma_i$ for $i = 1, 2, 3$. If there exists $i \in \{1, 2, 3\}$ s.t. $\Gamma_{j_i} \subseteq \sigma$, then let us define $\Psi_\sigma = \Theta_{j_i}$ and $\Delta_\sigma = \Gamma_{j_i}$, so that $CL_f(Sp) \vdash \Delta_\sigma \cup \Psi_\sigma \supset \varepsilon$ from condition 1 and $\Delta_\sigma \subseteq \sigma$ by construction.

  Otherwise $\Gamma_{j_1} = \Gamma'_{j_1} \cup \{D(t_{n+1})\}$, $\Gamma_{j_2} = \Gamma'_{j_2} \cup \{D(t'_{n+1})\}$ and $\Gamma_{j_3} = \Gamma_{j_3} \cup \{t_{n+1} = t'_{n+1}\}$; in this case let us define $\Delta_\sigma = \Gamma'_{j_1} \cup \Gamma'_{j_2} \cup \Gamma'_{j_3}$ and $\Psi_\sigma = \Theta_{j_1} \cup \Theta_{j_2} \cup \Theta_{j_3} \cup D(Var(t_{n+1} = t'_{n+1}))$; also in this case $CL_f(Sp) \vdash \Delta_\sigma \cup \Psi_\sigma \supset \varepsilon$, because of rule $7_f$ and condition 1, and $\Delta_\sigma \subseteq \sigma$ by construction.

  Thus the conditional formulas $\Delta_\sigma \cup \Psi_\sigma \supset \varepsilon$ for all sections $\sigma$ of $\mathscr{R}$ satisfy the condition 1 and condition 2 for $\mathscr{R} = \{R(t_i, t'_i) \mid i = 1, \ldots, n\}$. Therefore, because of the inductive hypothesis, there exist finite $\Theta' \subseteq \bigcup_{\sigma \in \mathrm{Sec}(\mathscr{R})} \Psi_\sigma$ and $Y' \subseteq Var(\bigcup_{\sigma \in \mathrm{Sec}(\mathscr{R})} \Delta_\sigma)$ s.t. $CL_f(Sp) \vdash D(Y') \cup \Theta' \supset \varepsilon$.

  Since both $Var(\bigcup_{\sigma \in \mathrm{Sec}(\mathscr{R})} \Delta_\sigma) \subseteq Var(\bigcup_{j=1,\ldots,k} \Gamma_j)$ and $\bigcup_{\sigma \in \mathrm{Sec}(\mathscr{R})} \Psi_\sigma \subseteq \bigcup_{j=1,\ldots,k} \Theta_j \cup Var(t_{n+1} = t'_{n+1})$, we have $CL_f(Sp) \vdash D(Y) \cup \Theta \supset \varepsilon$, where $\Theta = \Theta' - D(Var(t_{n+1} = t'_{n+1}))$ and $Y = Y' \cup (\Theta' - \Theta)$. □

**Lemma 4.18.** *If* $CL_f(Sp) \vdash \Theta_j \cup \Gamma_j \supset \varepsilon$ *for all* $j \in J$, $\Gamma = \{\Gamma_j \mid j \in J\}$ *is ininfluent and* $J$ *is finite, then* $CL_f(Sp) \vdash D(Y) \cup \Theta \supset \varepsilon$ *holds for some finite* $\Theta \subseteq \bigcup_{j \in J} \Theta_j$ *and* $Y \subseteq Var(\bigcup_{j \in J} \Gamma_j)$.

**Proof.** Because of Proposition 4.2, there exist $I = \{1, \ldots, k\}$ and terms $t_i, t'_i$ for $i \in I$ s.t. for any section $\sigma$ of $\mathscr{R} = \{R(t_i, t'_i) \mid i \in I\}$, where $R(t, t') = \{D(t), D(t'), t = t'\}$, there exists $j \in J$ s.t. $\Gamma_j \subseteq \sigma$. Thus we can apply Lemma 4.17. □

**Theorem 4.19.** *Let* $Sp = (\Sigma, Ax)$ *be a conditional specification s.t.* $\Delta$ *is finite for all* $\Delta \supset \varepsilon$ *in* $Ax$ *and* $X$ *be a family of variables.*

*Then the system* $CL_f(Sp)$ *is* seq-*complete for* $Sp$ *and* $X$.

**Proof.** The proof relies on the seq-completeness of $CL(Sp)$. We show by induction over the rules of $CL(Sp)$, that $CL(Sp) \vdash \Delta \supset \varepsilon$ implies that there exists a finite $\Gamma \subseteq \Delta$ s.t. $CL_f(Sp) \vdash \Gamma \supset \varepsilon$. Thus for any elementary formula $\varepsilon$ if $A \models D(X) \supset \varepsilon$ for all $A \in PMod(Sp)$, then $CL(Sp) \vdash D(X') \supset \varepsilon$ for some $X' \subseteq X$, because of the seq-completeness of $CL(Sp)$, and hence $CL_f(Sp) \vdash D(Y) \supset \varepsilon$ for a finite $Y \subseteq X' \subseteq X$. Let us give the inductive proof.

The proper axioms and the axioms $1, \dots, 4$ are common to both the systems and have finitary premises, so that the thesis trivially follows.

Let us consider rule 5, assume that its premises have been deduced, i.e. that $CL(Sp) \vdash \Delta \cup \Gamma \supset \varepsilon$ and $CL(Sp) \vdash \Delta_\gamma \supset \gamma$ for any $\gamma \in \Gamma$, and show that there exists a finite

$$\Gamma'' \subseteq \left(\Delta \cup \bigcup_{\gamma \in \Gamma} \Delta_\gamma\right) \cup D(Var(\Gamma) - Var\left(\left(\Delta \cup \bigcup_{\gamma \in \Gamma} \Delta_\gamma\right) \supset \varepsilon\right)$$

s.t. $CL_f(Sp) \vdash \Gamma'' \supset \varepsilon$.

Because of the inductive hypothesis, $CL_f(Sp) \vdash \Delta' \cup \Gamma' \supset \varepsilon$ for some finite $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$. Let $\Gamma'$ be $\{\gamma_1, \dots, \gamma_n\}$; because of the inductive hypothesis, $CL_f(Sp) \vdash \Delta'_{\gamma_i} \supset \gamma_i$ for $i = 1, \dots, n$ and some finite $\Delta'_{\gamma_i} \subseteq \Delta_{\gamma_i}$. It is easy to check that applying $n$ times the rule $5_f$ we can replace once at a time the $\gamma_i$ by $\Delta'_{\gamma_i}$ and then, always by rule $5_f$, we can get rid of the superfluous variables (i.e. variables of $\gamma_i$ already occurring in $\Delta_{\gamma_i}$) so that

$$CL_f(Sp) \vdash \left(\Delta' \cup \bigcup_{i=1,\dots,n} \Delta'_{\gamma_i}\right) \cup D(Var(\Gamma') - Var\left(\left(\Delta' \cup \bigcup_{i=1,\dots,n} \Delta'_{\gamma_i}\right) \supset \varepsilon\right) \supset \varepsilon$$

and hence we have the thesis for

$$\Gamma'' = \left(\Delta' \cup \bigcup_{i=1,\dots,n} \Delta'_{\gamma_i}\right) \cup D\left(Var(\Gamma') - Var\left(\Delta' \cup \bigcup_{i=1,\dots,n} \Delta'_{\gamma_i}\right)\right).$$

Let us consider rule 6, assume that its premise has been deduced, i.e. that $CL(Sp) \vdash \Delta \supset \varepsilon$, and show that there exists a finite

$$\Gamma \subseteq \{D(t_x) \mid x \in X_s, s \in S\} \cup \{\delta[t_x/x \mid x \in X_s, s \in S] \mid \delta \in \Delta\}$$

s.t. $CL_f(Sp) \vdash \Gamma \supset \varepsilon$. Because of the inductive hypothesis, $CL_f(Sp) \vdash \Gamma' \supset \varepsilon$ for some finite $\Gamma' \subseteq \Delta$. It is easy to check that applying $n$ times the rule $5_f$, starting from $\Gamma' \supset \varepsilon$, we have $CL_f(Sp) \vdash \{D(t_{x_1}), \dots, D(t_{x_n})\} \cup \{\gamma[t_{x_1}/x_1, \dots, t_{x_n}/x_n] \mid \gamma \in \Gamma'\} \supset \varepsilon[t_{x_1}/x_1, \dots, t_{x_n}/x_n]$, for $\{x_1, \dots, x_n\}$ the variables of $X$ appearing in $\Gamma' \supset \varepsilon$.

Thus

$$CL_f(Sp) \vdash \{D(t_{x_1}), \dots, D(t_{x_n})\} \cup \{\gamma[t_x/x \mid x \in X] \mid \gamma \in \Gamma'\} \supset \varepsilon[t_x/x \mid x \in X]$$

and $\Gamma = \{D(t_{x_1}), \ldots, D(t_{x_n})\} \cup \{\gamma[t_x/x \mid x \in X] \mid \gamma \in \Gamma'\}$ is a finite subset of $\{D(t_x) \mid x \in X_s, s \in S\} \cup \{\delta[t_x/x \mid x \in X_s, s \in S] \mid \delta \in \Delta\}$, because $\{x_1, \ldots, x_n\} \subseteq X$, and $\Gamma' \subseteq \Delta$ is finite.

Thus the only nontrivial step is the elimination rule. Let us assume that $CL(Sp) \vdash \Theta_j \cup \Gamma_j \supset \varepsilon$ for all $j \in J$, that $\Gamma = \{\Gamma_j \mid j \in J\}$ is ininfluent and show that $CL_f(Sp) \vdash D(Y) \cup \Theta \supset \varepsilon$ for some $\Theta \subseteq \bigcup_{j \in J} \Theta_j$ and $Y \subseteq Var(\bigcup_{j \in J} \Gamma_j)$. Because of the inductive hypothesis, $CL_f(Sp) \vdash \Theta'_j \cup \Gamma'_j \supset \varepsilon$ follows from $CL(Sp) \vdash \Theta_j \cup \Gamma_j \supset \varepsilon$, where both $\Theta'_j \subseteq \Theta_j$ and $\Gamma'_j \subseteq \Gamma_j$ are finite, for all $j \in J$.

Moreover, $\Gamma' = \{\Gamma'_j \mid j \in J\}$ is ininfluent, because $\Gamma'_j \subseteq \Gamma_j$ implies $\mathsf{Sec}(\Gamma') \subseteq \mathsf{Sec}(\Gamma)$. Thus, because of Lemma 4.16 for $\mathsf{Val} = \mathrm{EForm}(\Sigma, X)$ and $\mathscr{R} = \{\{D(t), D(t'), t = t'\} \mid t, t' \in T_\Sigma(Var)\}$, there exists a finite $I \subseteq J$ s.t. $\{\Gamma'_i \mid i \in I\}$ is ininfluent and hence, because of Lemma 4.18, we have $CL_f(Sp) \vdash D(Y) \cup \Theta \supset \varepsilon$ for some $\Theta \subseteq \bigcup_{i \in I} \Theta'_i$ and $Y \subseteq Var(\bigcup_{i \in I} \Gamma'_i)$. As $I$ and both $\Gamma'_i$ and $\Theta'_i$ for any $i \in I$ are finite, also $\Theta$ and $Y$ are finite; moreover $\Theta \subseteq \bigcup_{i \in I} \Theta'_i \subseteq \bigcup_{j \in J} \Theta'_j \subseteq \bigcup_{j \in J} \Theta_j$ and $Y \subseteq Var(\bigcup_{i \in I} \Gamma'_i) \subseteq Var(\bigcup_{j \in J} \Gamma_j)$ so that we have the thesis. $\square$

## 5. Conclusions

Motivated by the higher-order specifications of partial term-generated models, in this paper we have fully explored and solved the problem of the existence of initial and free models for partial conditional specifications and the related problem of equational deduction. Note that the existence problem is undecidable even in the finitary case and thus we may only look for sufficient effective conditions guaranteeing the existence. Since our results can be easily instantiated to give the corresponding known results for the positive conditional and the total case, it seems to us that our presentation has also some interest as a unifying framework. As we have shown in [1], the results apply to higher-order specifications whenever we consider classes of term-extensional (and term-generated extensional) models. Moreover in [2, 3] we have shown that, via skolemization, we can convert a (finitary) conditional higher-order specification $Sp$ to a (finitary) first-order partial conditional specification $SSp$, whose models satisfy the same (open) equations as the models of $Sp$. Thus $CL(SSp)$ (or $CL_f(SSp)$, if $Sp$ is finitary) is equationally complete w.r.t. the extensional models of $Sp$.

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